Dissertation

The p-adic valuations of the critical values of L-functions associated to elliptic curves

Author
Keiichiro Nomoto

Supervisor
Shinichi Kobayashi

Kyushu University

Graduate School of Mathematics

Contents

Acknowledgements

First of all, I would like to express my sincere gratitude to my supervisor Shinichi Kobayashi for his innumerably valuable advice throughout the course of my study. Without his thoughtful guidance, I would not have been able to enjoy mathematics so deeply, nor would I have been able to complete this thesis. I am also grateful to Shun'ichi Yokoyama at Tokyo Metropolitan University for his helpful advice on Computer Algebra and for further showing me the joy of mathematics using computers.

In addition, I would like to thank to Taiga Adachi, Akihiro Goto, Takeshi Kawano, Satoshi Kumabe, and Ryota Shii for supporting me privately. Meeting them has been an invaluable asset to me.

Finally, I would like to express my greatest gratitude to Minami Okubo. We started dating just before I entered the doctoral course, and I spent all three years of the course with her. I was able to get through the doctoral course without any major setbacks because she was always by my side and kept giving me kind words. Without her, I would not be who I am now. I am grateful to her - my dearest, all the time.

Abstract

The algebraic part of the critical value of an L-function associated to an elliptic curve is the algebraic number obtained by dividing the critical value by an appropriate period of the elliptic curve. In general, a critical value of an L-function is conjectured to be transcendental. However, by taking out its algebraic part, it is possible to examine the critical value of the L-function algebraically and obtain more precise information. The Birch and Swinnerton–Dyer (BSD) conjecture asserts that the algebraic part of the critical value of the Hasse–Weil L-function is described by arithmetic invariants of an elliptic curve such as the Tate–Shafarevich group and Tamagawa factors. Therefore, studying the algebraic part of the critical value is deeply related to calculating these invariants. In this thesis, we discuss two topics related to the p-adic valuations of the algebraic parts of the critical values in part I and part II respectively.

In part I, we study the behavior of the 2-adic valuations of the critical values for elliptic curves with complex multiplication. In 1997, Zhao [Zha97] has given a lower bound of the 2-adic valuations of the central values of the Hecke L-functions associated to a certain family of elliptic curves with complex multiplication by the ring of Gaussian integers indexed by a square-free Gaussian integer D. His method is based on the number of the primes dividing D, and is sometimes referred to as *Zhao's method*. To date, Zhao's method has been applied to various families of elliptic curves and has even been devised as an application that shows non-vanishing of critical values of L-functions associated to elliptic curves, making it one of the most promising methods for future development. However, due to technical reasons, Zhao's method has been applied only when the indices of the primes dividing the parameter D are all equal. In this study, we overcome this problem for a certain family of CM elliptic curves and have succeeded in removing the condition on the indices of the primes. In the proof, multiple use of Zhao's method is essential. It is expected that this method will make it possible to give a lower bound of the p-adic valuations of the central values of the Hecke L-functions associated with all CM elliptic curves defined over an imaginary quadratic field with class number one.

In part II, we give a certain answer to the problem of determining the ranks of two elliptic curves defined over the field of rational numbers. Let p denote a prime number. Rodríguez-Villegas and Zagier [RZ95] have given a necessary and sufficient condition that the rank of the elliptic curve $A_p : x^3 + y^3 = p$ is equal to 2 by the constant term of a polynomial defined by a simple recurrence formula. There are two main results of this study. One is that we have given another formula that is more efficient than the one they gave in some sense. The other is that we have newly given a necessary and sufficient condition that the rank of the elliptic curve $E_{-p} : y^2 = x^3 + px$ is equal to 2 by using a simple recurrence formula. One of the key points of the proof is to derive a congruence relation modulo p between the algebraic part of the critical value of the Hasse–Weil L-function of the elliptic curve A_p (resp. E_{-p}) depending on the prime number p and that of the central value of some Hecke L-function associated to the prime-independent elliptic curve $A_1 : x^3 + y^3 = 1$ (resp. $E_{-1} : y^2 = x^3 + x$). By using this result, the computation of the ranks of these elliptic curves is reduced to a naive computation of polynomials, which can be easily implemented using a computer.

Part I

The 2-adic valuations

Chapter 1

Introduction

1.1 Background

First, let us briefly explain the algebraic part of the critical value of the Hasse–Weil L-function for an elliptic curve. Let E be an elliptic curve defined over a number field K and $L(E/K,s)$ the Hasse–Weil L-function, conjectured to have an analytic continuation to the entire complex plane. For each elliptic curve, a value called period is defined up to multiplication by a non-zero algebraic number. For example, the value obtained by integrating an invariant differential of an elliptic curve over some domain is a period. The detailed definition of a period is given in Section 2.1, and here we fix a suitable period $\Omega_{E/K}$ of E. There is a deep relationship between the period $\Omega_{E/K}$ and the critical value of $L(E/K,s)$. We assume that $L(E/K,s)$ has an analytic continuation. Then, the value

$$\frac{L(E/K,1)}{\Omega_{E/K}} \tag{1.1}$$

is expected to be algebraic. We call such a value the *algebraic part* of the critical value of $L(E/K,s)$. Note that the algebraic part depends on the choice of a period.

The Birch and Swinnerton–Dyer (BSD) conjecture asserts that if $L(E/K,1)\neq 0$, then the algebraic part (1.1) can be written in terms of arithmetic invariants of the elliptic curve E, that is, the following equation, both sides of which lie in $\overline{\mathbb{Q}}$, holds:

$$\frac{L(E/K,1)}{\Omega_{E/K}} = \frac{\prod_{\mathfrak{p}} c_{\mathfrak{p}} \cdot \#\text{Ш}(E/K)}{\sqrt{|d_K|}\cdot(\#E(K)_{\text{tors}})^2}, \tag{1.2}$$

where d_K is the discriminant of K, $c_{\mathfrak{p}}$ is the Tamagawa factor at the prime $\mathfrak{p}$ and Ш(E/K) is the Tate–Shafarevich group. The equality for the p-adic valuation of both sides of equation (1.2) for each rational prime p is called the *p-part of the BSD conjecture*, which has also not been completely proven. Hence, studying the p-adic valuations of the algebraic parts is important.

We are especially concerned with the 2-adic valuations of the algebraic parts. There are two main reasons for focusing on $p=2$. The first is that $p=2$ is difficult to deal with in Iwasawa theory. A common approach to the p-part of the BSD conjecture is now to prove the Iwasawa main conjecture for elliptic curves. However currently, $p=2$ is somewhat difficult to handle in Iwasawa theory. In fact, results of Kato [Kat04], Skinner–Urban [SU14], Rubin [Rub91] and others have been treated only for sufficiently large primes. Therefore, the case of $p=2$ remains as an exception. The second is the algebraic part seems to be mostly the 2-part. This is because the Tamagawa factor $c_{\mathfrak{p}}$ takes integer values between 1 and 4 if an elliptic curve has additive or split multiplicative reduction at $\mathfrak{p}$.

In part I, we consider elliptic curves defined over $\mathbb{Q}(i)$ with complex multiplication by $\mathbb{Z}[i]$. In this case, the p-adic valuation of the algebraic part makes sense for each prime number p from the following theorems.

Theorem 1.1 (Hecke–Deuring)**.** Let E be an elliptic curve defined over a number field F with complex multiplication by the ring of integers $\mathcal{O}_K$ of an imaginary quadratic field K.

(i) Assume that $K \subset F$ and we write $\psi_{E/F}$ be the Hecke character associated to E/F. Then

$$L(E/F, s) = L(\psi_{E/F}, s)L(\overline{\psi_{E/F}}, s).$$

(ii) Assume that $K \not\subset F$, and let $F' = FK$. We write $\psi_{E/F'}$ be the Hecke character associated to E/F'. Then

$$L(E/F, s) = L(\psi_{E/F'}, s).$$

In particular, $L(E/F, s)$ has an analytic continuation to the entire complex plane.

Proof. For example, see [Sil94, CHAPTER II, Theorem 10.5] and [Sil94, CHAPTER II, Corollary 10.5.1]. □

Theorem 1.2 (Damerell's Theorem)**.** Let E be an elliptic curve defined over an imaginary quadratic field K with complex multiplication by $\mathcal{O}_K$. We suppose the class number of K is equal to one. Let $\mathcal{L}$ be the period lattice of E and take $\Omega_E \in \mathbb{C}^\times$ so that $\mathcal{L} = \Omega_E\mathcal{O}_K$. Denote the Hecke character of K associated to E by ψ. For each positive integer k, we have

$$\frac{L(\overline{\psi}^k, k)}{\Omega_E^k} \in \overline{\mathbb{Q}}.$$

Proof. For example, see [Dam70, Theorem 1] or [Rub99, Corollary 7.18]. □

Remark 1.3. When the class number of K is not necessarily one in Theorem 1.2, the algebraicity of the algebraic part follows, for example, from [GS81, Theorem 7.1].

Remark 1.4. As an aside, when E is defined over $\mathbb{Q}$ which does not necessarily have complex multiplication, the results of Wiles and others ([Wil95], [TW95], [Bre+01]) show the analytic continuation of $L(E/\mathbb{Q}, s)$. Furthermore, it follows from the results of Manin [Man72] and Drinfel'd [Dri73] that the algebraic part of $L(E/\mathbb{Q}, 1)$ is indeed algebraic. The algebraic part of the special value at $s = 1$ of the derivative of $L(E/\mathbb{Q}, s)$ is also defined with a slight modification, and the algebraicity in that case is proved by Gross and Zagier [GZ86].

1.2 Previous research and main result

In 1997, Zhao [Zha97] has given a lower bound of the 2-adic valuations of the central values of the Hecke L-functions associated to a certain family of elliptic curves $E_D : y^2 = x^3 - Dx$ defined over $\mathbb{Q}(i)$ with $D \in \mathbb{Z}[i]$ square-free. His method is based on the number of the primes dividing D, and is sometimes referred to as *Zhao's method*.

Several works are giving lower bounds of the p-adic valuations of various families of elliptic curves with complex multiplication when $p = 2, 3$, using Zhao's method. First, we give some results for elliptic curves of the form $y^2 = x^3 - Dx$. Zhao has given a lower bound of the 2-adic valuations when $D = (\pi_1 \cdots \pi_n)^2 \in \mathbb{Z}[i]$ ($\pi_i \equiv 1 \bmod 4$) is the square of the product of distinct Gaussian primes in [Zha97] and when $D = (p_1 \cdots p_n)^2$ ($p_i \equiv 1 \bmod 8$) is the square of the product of distinct rational primes in [Zha01]. He has also given it for $D = 4(\pi_1 \cdots \pi_n)^2$ ($\pi_i \equiv 1 \bmod 2+2i$) in [Zha03]. Qiu and Zhang [QZ02a] have given a lower bound of the 2-adic valuations for $D = \pi_1 \cdots \pi_n$, $(\pi_1 \cdots \pi_r)^2 \pi_{r+1} \cdots \pi_n$ ($\pi_i \equiv 1 \bmod 4$). In the latter case $D = (\pi_1 \cdots \pi_r)^2 \pi_{r+1} \cdots \pi_n$, however, no proof has been given. Next, we give some results for elliptic curves of the form $y^2 = x^3 - 432D$. Qiu and Zhang [QZ02b] have given a lower bound of the 3-adic valuations when $D = (\pi_1 \cdots \pi_n)^2 \in \mathbb{Z}[\omega]$ ($\pi_i \equiv 1 \bmod 6$) is the square of the product of distinct Eisenstein primes. Qiu [Qiu03] also has given it for $D = (\pi_1 \cdots \pi_n)^4$ ($\pi_i \equiv 1 \bmod 6$) and for $D = (\pi_1 \cdots \pi_n)^3$ ($\pi_i \equiv 1 \bmod 12$). Kezuka [Kez21] has given a lower bound of the 3-adic valuations for the elliptic curves $y^2 = x^3 - 432D^2$ defined over $\mathbb{Q}$ when D is a cube-free integer with $(D, 3) = 1$. There are also some studies on CM elliptic curves with these j-invariants being not 0 or 1728 (*cf.* [Coa+15], [Coa+14], [Cho19]).

Let $K = \mathbb{Q}(i)$. We consider the elliptic curve $E_{-D} : y^2 = x^3 + Dx$ defined over K for $D \in K$ which is coprime to 2. We write the Hecke character associated to E_{-D} as ψ_{-D}. We give a lower bound for the 2-adic valuation of the algebraic part of $L(\overline{\psi_{-D}}, 1)$. The following theorem is the main result.

Theorem 1.5. Suppose $D \in \mathcal{O}_K$, quartic-free, and is congruent to 1 modulo $2+2i$. Let ψ_{-D} be the Hecke character associated to the elliptic curve $E_{-D} : y^2 = x^3 + Dx$ defined over K. We define $L_2(\overline{\psi_{-D}}, s)$ to be the Hecke L-function of $\overline{\psi_{-D}}$ omitting the Euler factor corresponding to the prime $(1+i)\mathcal{O}_K$. If $D \notin K^{\times 2}$, then we have

$$v_2\left(\frac{L_2(\overline{\psi_{-D}}, 1)}{\Omega}\right) \geq \frac{r(D) - 2}{2},$$

where $r(D)$ is the number of distinct primes dividing D, $\Omega = 2.6220575\ldots$ is the least positive real element of the period lattice of $E_1 : y^2 = x^3 - x$ and v_2 is the 2-adic valuation of $\overline{\mathbb{Q}_2}$ normalized so that $v_2(2) = 1$.

Remark 1.6. When $D \in K^{\times 2}$, Zhao has given the lower bound $(2r(D) - 3)/2$ [Zha03, Theorem 1]. Note that Zhao uses a period of E_{4D}, while we use a period Ω of E_1.

Remark 1.7. The condition that $D \in \mathcal{O}_K$, quartic-free and congruent to 1 modulo $2+2i$ in Theorem 1.5 is not essential. If $D \in K$ is not quartic-free, then we can take $D_0 \in \mathcal{O}_K$ so that it is quartic-free and E_{-D} is isomorphic to E_{-D_0} over K. For any $D \in \mathcal{O}_K$ which is coprime to 2, only one of $\{\pm D, \pm iD\}$ is congruent to 1 modulo $2+2i$. For more details, see Section 2.2.

Remark 1.8. The lower bound of Theorem 1.5 is expected to be sharp in the sense that there exist elliptic curves E_{-D} for which equality holds. See the numerical examples in Section 3.3.

We prove Theorem 1.5 combining Theorem 3.4 with Theorem 3.5. Here, Theorem 3.4 deals with the case where all the indices of the primes dividing D are equal, and Theorem 3.5 deals with the other case. The key of the proof of Theorem 3.4 and Theorem 3.5 is to consider not only an elliptic curve E_{-D} for a parameter D but also elliptic curves E_{-D_T} for all divisors D_T of D. Theorem 3.4 is proved by Zhao's method, that is, using the induction on the number of the primes dividing D. However, due to technical reasons, Zhao's method can only be applied to the case where all the indices of the primes dividing D are equal. In order to apply Zhao's method to the other case, we decompose D into $D_1 D_2 D_3$, where D_i is the product of the primes dividing D whose indices are all equal to i. By iterating Zhao's method for each D_i, we give a lower bound of the 2-adic valuation for general D and prove Theorem 3.5.

We deal only with the family $E_{-D} : y^2 = x^3 + Dx$ in this paper. However, the essence of the proof of Theorem 3.5 is that D can be uniquely decomposed into the product of primes in K. Therefore, our iterative Zhao's method may be applicable to all CM elliptic curves defined over imaginary quadratic fields with class number one.

After writing our paper [Nom22a] stating Theorem 1.5, we noticed that Kezuka has also given a lower bound of the 3-adic valuations for the elliptic curves $y^2 = x^3 - 432D^2$ defined over $\mathbb{Q}$ using an iterative Zhao's method similar to ours in the proof of [Kez21, Theorem 2.4].

Part I is organized as follows. In chapter 2, we make some calculations on various invariants of the elliptic curve E_{-D} and write the L-value at $s = 1$ as a finite sum using a special value of the Weierstrass $\wp$-function. In chapter 3, we give a lower bound of the 2-adic valuation of the L-value by using Zhao's method. For reference, numerical examples of a lower bound in Theorem 1.5 are included at the end of part I. In the proof of Theorem 3.5, we use Zhao's method iteratively. For this reason, the proof is complicated, and please refer to the inserted figures as necessary.

Chapter 2

Preliminaries

2.1 BSD invariants

In this section, we make some calculations on various invariants of the elliptic curve $E_{-D} : y^2 = x^3 + Dx$ defined over $K = \mathbb{Q}(i)$. Since E_{-D} is isomorphic to E_{-d^4D} over K for $d \in K^\times$, we may assume that $D \in \mathcal{O}_K$ and quartic-free. In the rest of part I, we consider only the elliptic curve $E_{-D} : y^2 = x^3 + Dx$ defined over K for $D \in \mathcal{O}_K$ that is coprime to 2 and quartic-free.

Proposition 2.1. The following holds:

$$\#E_{-D}(K)_{\text{tors}} = \begin{cases} 4 & (D \in K^{\times 2}), \\ 10 & (D = \pm(1 \pm 2i)), \\ 2 & (\text{otherwise}). \end{cases}$$

Proof. It is straightforward to verify the claim using the Nagell–Lutz theorem for K. In particular when $D \notin K^{\times 2}$, it is computed in [OS21, Lemma 6.2] and [OS21, Remark 6.3]. □

Proposition 2.2. Suppose $D \in \mathcal{O}_K$ is congruent to 1 modulo $2 + 2i$. The elliptic curve E_{-D} has bad reduction at all primes dividing $D\mathcal{O}_K$. Moreover, E_{-D} has good reduction at the prime $(1 + i)\mathcal{O}_K$ if and only if $(i/D)_4 = i$, where $(\cdot/\cdot)_4$ is the quartic residue character.

Proof. Since the discriminant of the equation $y^2 = x^3 + Dx$ is $(1 + i)^{12} D^3$ and D is quartic-free, the elliptic curve E_{-D} is minimal at all primes dividing $D\mathcal{O}_K$. Therefore, the first claim follows. We show that E_{-D} has good reduction at $(1 + i)\mathcal{O}_K$ when $(i/D)_4 = i$ using Tate's algorithm. In the other cases, we can show similarly that E_{-D} has bad reduction at $(1 + i)\mathcal{O}_K$. From now on, we follow Silverman's notation and steps [Sil94, p.366].

We start from step 1. Set $\pi = 1 + i$ and we have

$$\Delta = \pi^{12} D^3, \quad a_1 = a_2 = a_3 = a_6 = 0, \quad a_4 = D,$$
$$b_2 = b_6 = 0, \quad b_4 = 2D, \quad b_8 = -D^2.$$

Since $\pi \mid \Delta$, we proceed to Step 2. The curve $\widetilde{E}$ obtained by reduction of E at π has the singular point $(1, 0)$. Thus, we do the transformation $x \mapsto x + 1$ and obtain the new equation

$$y^2 = x^3 + 3x^2 + (D + 3)x + (D + 1)$$

whose reduction curve has the singular point $(0, 0)$. Then, we have

$$a_1 = a_3 = 0, \quad a_2 = 3, \quad a_4 = D + 3, \quad a_6 = D + 1,$$

$$b_2 = 12, \quad b_4 = 2D + 6, \quad b_6 = 4D + 4, \quad b_8 = -D^2 + 6D + 3.$$

We can easily check $\pi \mid b_2, \pi^2 \mid a_6, \pi^3 \mid b_6, b_8$ and proceed to Step 6. Let k be the residue field $\mathcal{O}_K/(\pi)$ and fix an algebraic closure $\overline{k}$. For simplcity, we set $a_{i,r} = \pi^{-r}a_i$. The following equations over k

$$Y^2 + a_1Y - a_2 \equiv (Y - \alpha)^2 \bmod \pi,$$
$$Y^2 + a_{3,1}Y - a_{6,2} \equiv (Y - \beta)^2 \bmod \pi$$

have the solution $\alpha = \beta = 1$. Thus, we do the transformation $y \mapsto y + x + \pi$ and obtain the new equation

$$y^2 + 2xy + 2\pi y = x^3 + 2x^2 + (D + 3 - 2\pi)x + (D + 1 - \pi^2).$$

Then, we have

$$a_1 = a_2 = 2, \quad a_3 = 2\pi, \quad a_4 = D + 3 - 2\pi, \quad a_6 = D + 1 - \pi^2,$$
$$b_2 = 12, \quad b_4 = 2D + 6, \quad b_6 = 4D + 4, \quad b_8 = -D^2 + 6D + 3.$$

We consider the factorization over $\overline{k}$ of the polynomial

$$P(T) = T^3 + a_{2,1}T^2 + a_{4,2}T + a_{6,3}.$$

If we write $D = 1 + (2 + 2i)(s + ti)$ for $s, t \in \mathbb{Z}$, then we see that $P(T) = T^3 - (s - t - 1)$. By properties of the quartic residue symbol, $(i/D)_4 = i$ is equivalent to $s - t \equiv 3 \bmod 4$. Thus, $P(T)$ has the triple root $T = 0$ and we proceed to Step 8. Since the polynomial over $\overline{k}$

$$Y^2 + a_{3,2}Y - a_{6,4} = Y^2 - s$$

has the double root $Y = 0$ if $s \equiv 0 \bmod 2$ and $Y = 1$ if $s \equiv 1 \bmod 2$. We suppose $s \equiv 0 \bmod 2$ and proceed to Step 9. (For the case $s \equiv 1 \bmod 2$, we proceed to Step 9 after transformation $y \mapsto y + \pi^2$.) Since $\pi^4 \mid a_4$ and $\pi^6 \mid a_6$, we proceed to Step 11. Then, the transformation $x \mapsto \pi^2 x, y \mapsto \pi^3 y$ leads to the new equation

$$y^2 + \frac{2}{\pi}xy + \frac{2}{\pi^2}y = x^3 + \frac{2}{\pi^2}x^2 + \frac{D + 3 - 2\pi}{\pi^4}x + \frac{D + 1 - \pi^2}{\pi^6}$$

whose discriminant is D^3. Therefore, the elliptic curve E has good reduction at $(1 + i)\mathcal{O}_K$ and we finish Tate's algorithm. □

Remark 2.3. If $(i/D)_4 = i$, then the minimal model of E_{-D} at $(1 + i)\mathcal{O}_K$ is

$$\begin{cases} y^2 + (1 - i)xy - iy = x^3 - ix^2 - \frac{D+1-2i}{4}x + \frac{iD+2+i}{8} & (s \equiv 0 \bmod 2), \\ y^2 + (1 - i)xy + (1 - 2i)y = x^3 - ix^2 - \frac{D+1-6i}{4}x + \frac{iD+6+9i}{8} & (s \equiv 1 \bmod 2), \end{cases}$$

where $D = 1 + (2 + 2i)(s + ti)$ $(s, t \in \mathbb{Z})$.

Local informations at $(1+i)\mathcal{O}_K$ including Kodaira symbols are summarized in Table 2.1. For other primes that divide D, we obtain Table 2.2 by Tate's algorithm. Here, D_i is the product of the primes dividing D whose indices are all equal to i.

$(i/D)_4$	Kodaira Symbol	m	v	f	c
± 1	I_0^*	5	12	8	2
i	I_0	1	0	0	1
$-i$	II^*	9	12	4	1

Table 2.1: Local informations at $(1+i)\mathcal{O}_K$

$\pi \mid D$	Kodaira Symbol	m	v	f	c
$\pi \mid D_1$	III	2	3	2	2
$\pi \mid D_2,\ (D_1D_3/\pi)_2 = 1$	I_0^*	5	6	2	4
$\pi \mid D_2,\ (D_1D_3/\pi)_2 = -1$	I_0^*	5	6	2	2
$\pi \mid D_3$	III^*	8	9	2	2

Table 2.2: Local informations at $\pi\mathcal{O}_K$ $(\pi \mid D)$

Next, we recall the definition of the period of an elliptic curve appearing in the BSD conjecture. For details, see [Tat95] or [DD10] for example. Let E be an elliptic curve defined over a number field F and fix an invariant differential ω on E. Denote the normalized absolute value at a place v of F by $|\cdot|_v$. Let ω_v^o be a Néron differential at a finite place v. Then, we define

$$\Omega_{E/F} := \prod_{v\nmid\infty} \left|\frac{\omega}{\omega_v^o}\right|_v \prod_{\substack{v\mid\infty \\ \text{real}}} \int_{E(F_v)} |\omega| \prod_{\substack{v\mid\infty \\ \text{complex}}} 2\int_{E(F_v)} \omega\wedge\overline{\omega},$$

where F_v is the completion of F at v. Note that $\Omega_{E/F}$ is independent of the choice of ω by the product formula and the choice of ω_v^o. If we fix a Weierstrass model of E with discriminant $\Delta_{E/F}$, in terms of the minimal discriminant ideal $\mathfrak{d}_{E/F}$, the period $\Omega_{E/F}$ is rewritten as follows:

$$\Omega_{E/F} = \left|\frac{N(\Delta_{E/F})}{N(\mathfrak{d}_{E/F})}\right|^{1/12} \prod_{\substack{v\mid\infty \\ \text{real}}} \int_{E(F_v)} |\omega| \prod_{\substack{v\mid\infty \\ \text{complex}}} 2\int_{E(F_v)} \omega\wedge\overline{\omega}.$$

Let $\omega_1 = dx/2y$ be an invariant differntial of $E_1 : y^2 = x^3 - x$ and $E_1^0(\mathbb{R})$ the connected component of $E_1(\mathbb{R})$ containing the identity of E_1. Then, the period lattice of ω_1 is of the form $\Omega\mathbb{Z} + i\Omega\mathbb{Z}$, where

$$\Omega := \int_{E_1^0(\mathbb{R})} \omega_1 = \int_1^\infty \frac{dx}{\sqrt{x^3 - x}} \fallingdotseq 2.6220576.$$

Note that $\int_{E_1(\mathbb{C})} \omega_1 \wedge \overline{\omega_1}$ is equal to the area of the fundamental parallelogram of the lattice $\Omega\mathbb{Z} + i\Omega\mathbb{Z}$ and therefore equal to Ω^2.

Proposition 2.4. We have

$$\Omega_{E_{-D}/K} = \begin{cases} \dfrac{4\Omega^2}{N(D)^{1/4}} & ((i/D)_4 = i), \\ \dfrac{2\Omega^2}{N(D)^{1/4}} & (\text{otherwise}). \end{cases}$$

Proof. Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and a quartic root $(-D)^{1/4} \in \mathbb{C}$. We write $E_{-D} : y^2 = x^3 + Dx$ and $E_1 : Y^2 = X^3 - X$. Let $\omega_{-D} = dx/2y$ an invariant differential on E_{-D}. The transformaion $x = (-D)^{2/4}X$, $y = (-D)^{3/4}Y$ leads to an isomorphism over $\mathbb{C}$ between E_{-D} and E_1. Therefore, we obtain

$$\int_{E_{-D}(\mathbb{C})} \omega_{-D} \wedge \overline{\omega_{-D}} = N(D)^{-1/4} \int_{E_1(\mathbb{C})} \omega_1 \wedge \overline{\omega_1} = N(D)^{-1/4}\Omega^2.$$

By Table 2.1 and Table 2.2, we have

$$\left| \frac{N(\Delta_{E/K})}{N(\mathfrak{d}_{E/K})} \right|^{1/12} = \begin{cases} 2 & ((i/D)_4 = i), \\ 1 & (\text{otherwise}). \end{cases}$$

Thus, the proposition follows. □

2.2 L-value as a finite sum

In this section, we write the L-value at $s = 1$ as a finite sum using a special value of the Weierstrass $\wp$-function. Theorem 2.10 has already been proved by Birch and Swinnerton–Dyer [BS65]; however, for readers convenience, we calculate it again.

Let ψ_{-D} be the Hecke character of K associated to E_{-D} and let $\Omega = \int_1^\infty dx/\sqrt{x^3 - x}$ be a period of $E_1 : y^2 = x^3 - x$. For a non-zero element $g \in \mathcal{O}_K$, $L_g(\overline{\psi}, s)$ denotes the Hecke L-function of $\overline{\psi}$ omitting all Euler factors corresponding to the primes that divide $g\mathcal{O}_K$; that is;

$$L_g(\overline{\psi}, s) = L(\overline{\psi}, s) \prod_{\mathfrak{p} | g\mathcal{O}_K} \left(1 - \frac{\overline{\psi}(\mathfrak{p})}{N\mathfrak{p}^s}\right).$$

For a non-zero ideal $\mathfrak{g}$ of $\mathcal{O}_K$, we define $L_{\mathfrak{g}}(\overline{\psi}, s)$ in the same way. Fix $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}_2}$ as algebraic closures of $\mathbb{Q}$ and $\mathbb{Q}_2$, and fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_2}$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let v_2 denote the 2-adic valuation of $\mathbb{Q}_2$ normalized so that $v_2(2) = 1$ and extend to $\overline{\mathbb{Q}_2}$, which is also written as v_2.

Proposition 2.5. Suppose $D \in \mathcal{O}_K$ is congruent to 1 modulo $2+2i$. Then, the following holds:

$$v_2\left(\frac{L_{2D}(\overline{\psi_{-D}}, 1)}{\Omega}\right) = v_2\left(\frac{L_2(\overline{\psi_{-D}}, 1)}{\Omega}\right) = \begin{cases} v_2\left(\frac{L(\overline{\psi_{-D}}, 1)}{\Omega}\right) - \frac{1}{2} & ((i/D)_4 = i), \\ v_2\left(\frac{L(\overline{\psi_{-D}}, 1)}{\Omega}\right) & (\text{otherwise}). \end{cases}$$

Proof. For each prime π dividing D, we have $\psi_{-D}((\pi)) = 0$, and $\psi_{-D}((1+i)) = 0$ if $(i/D)_4 \neq i$ by Proposition 2.2. When $(i/D)_4 = i$, $\psi_{-D}((1+i))$ is non-zero and equal to $u(1+i)$ for some $u \in \mathcal{O}_K^\times$. Therefore, we have

$$v_2\left(\frac{\psi_{-D}((1+i))}{N(1+i)}\right) = v_2\left(\frac{u(1+i)}{2}\right) = -\frac{1}{2} \neq 0.$$

Thus, the 2-adic valuation of the Euler factor at $(1+i)\mathcal{O}_K$ is equal to $-1/2$. □

As mentioned in Section 1.2, we iterate Zhao's method. For this purpose, we first decompose D uniquely up to units in $\mathcal{O}_K$ according to the index of a prime dividing D, such as $D_1^{(n)}D_2^{(m)}D_3^{(\ell)}$, where

$$D_1^{(n)} = \prod_{\pi_{1,i} \in S_1} \pi_{1,i}, \quad D_2^{(m)} = \prod_{\pi_{2,j} \in S_2} \pi_{2,j}^2, \quad D_3^{(\ell)} = \prod_{\pi_{3,k} \in S_3} \pi_{3,k}^3,$$

and $S_1 = \{\pi_{1,1}, \dots, \pi_{1,n}\}, S_2 = \{\pi_{2,1}, \dots, \pi_{2,m}\}, S_3 = \{\pi_{3,1}, \dots, \pi_{3,\ell}\}$ are disjoint sets of distinct primes of $\mathcal{O}_K$ which are coprime to 2. Here, a prime of $\mathcal{O}_K$ is said to be primary if it is congruent to 1 modulo $2+2i$. For a prime π which is coprime to 2, it is known that only one of $\{\pm\pi, \pm i\pi\}$ is primary. Hence, all primes in S_i are assumed to be primary, and D is congruent to 1 modulo $2+2i$. We abbreviate $D_i^{(*)}$ as D_i if we do not care about the number of the primes in S_i.

Next, we represent all divisors D_T of D as follows. Let $T_1 \subset \{1, \dots, n\}, T_2 \subset \{1, \dots, m\}, T_3 \subset \{1, \dots, \ell\}$ be arbitrary subsets (including the case where T_1, T_2, and T_3 are empty sets). Then, we define

$$D_{T_1} = \prod_{i \in T_1} \pi_{1,i}, \quad D_{T_2} = \prod_{j \in T_2} \pi_{2,j}^2, \quad D_{T_3} = \prod_{k \in T_3} \pi_{3,k}^3$$

and $D_T = D_{T_1} D_{T_2} D_{T_3}$. When $T_i = \varnothing$ $(i = 1, 2, 3)$, we define $D_{T_i} = 1$.

For a lattice $\mathcal{L}$ of $\mathbb{C}$ and integer $k \geq 0$, we define the holomorphic function on the domain $\mathrm{Re}(s) > 1 + k/2$ by

$$H_k(z, s, \mathcal{L}) = \sum_{w \in \mathcal{L}}{}' \frac{\overline{(z+w)}^k}{|z+w|^{2s}}.$$

Here, $\sum'$ implies that $w = -z$ is excluded if $z \in \mathcal{L}$. The function $s \mapsto H_k(z, s, \mathcal{L})$ has the analytic continuation to the entire complex s-plane if $k \geq 1$. We set

$$\mathcal{E}_1^*(z, \mathcal{L}) = H_1(z, 1, \mathcal{L}).$$

Proposition 2.6 ([GS81, Proposition 5.5])**.** Let E be an elliptic curve over an imaginary quadratic field K with complex multiplication by $\mathcal{O}_K$. Fix a Weierstrass model of E and take $\Omega_E \in \mathbb{C}^\times$ such that the period lattice of E is $\Omega_E \mathcal{O}_K$. We write ϕ as the Hecke character of K associated to E and suppose the conductor of ϕ divides a non-zero integral ideal $\mathfrak{g}$ of K. Let B be a minimal set consisting of ideals prime to $\mathfrak{g}$ such that

$$\mathrm{Gal}(K(E[\mathfrak{g}])/K) = \{\sigma_{\mathfrak{b}} \mid \mathfrak{b} \in B\},$$

where $\sigma_{\mathfrak{b}}$ is the Artin symbol corresponding to $\mathfrak{b}$. We take $\rho \in \Omega_E K^\times$ such that $\rho \Omega_E^{-1} \mathcal{O}_K = \mathfrak{g}^{-1}$. Then, for $k \geq 1$, the following holds:

$$\frac{\overline{\rho}^k}{|\rho|^{2s}} L_{\mathfrak{g}}(\overline{\phi}^k, s) = \sum_{\mathfrak{b} \in B} H_k(\phi(\mathfrak{b})\rho, s, \mathcal{L}).$$

For the moment, we take $\Delta \in \mathcal{O}_K$, which is congruent to 1 modulo $2+2i$, so that the conductor of ψ_{-D_T} divides $4\Delta\mathcal{O}_K$. Later, we explicitly define Δ (see the paragraph after Lemma 3.1).

Lemma 2.7. We apply Proposition 2.6 to $E = E_{-D_T}, \phi = \psi_{-D_T}, \mathfrak{g} = 4\Delta\mathcal{O}_K$. Then a set B can be taken as

$$B = \{(4c + \Delta), (4c + (1+2i)\Delta) \mid c \in \mathcal{C}\},$$

where $\mathcal{C}$ is a complete system of representatives of $(\mathcal{O}_K/\Delta\mathcal{O}_K)^\times$.

Proof. Since the conductor of $\overline{\psi_{-D_T}}$ divides $4\Delta\mathcal{O}_K$, [GS81, Lemma 4.7] shows that the field $K(E_{-D_T}[4\Delta])$ coincides with $K(4\Delta)$, the ray class field of K associated to the modulus $4\Delta\mathcal{O}_K$. Thus the following isomorphism via the Artin map holds:

$$\mathrm{Gal}(K(E_{-D_T}[4\Delta])/K) \simeq (\mathcal{O}_K/4\Delta\mathcal{O}_K)^\times/\mathcal{O}_K^\times.$$

Hence the cardinality of B must be equal to $2\cdot\#(\mathcal{O}_K/\Delta\mathcal{O}_K)^\times$. Therefore, it is sufficient to show that the Artin symbols corresponding to any two different elements in B are different from each other. We show that $\sigma_{(4c+\Delta)} \neq \sigma_{(4c'+\Delta)}$ for $c \neq c' \in \mathcal{C}$. Assume that $\sigma_{(4c+\Delta)} = \sigma_{(4c'+\Delta)}$. Then $4c+\Delta$ must be congruent to $4c'+\Delta$ modulo 4Δ. However, this implies that c and c' belong same equivalence class in $(\mathcal{O}_K/\Delta\mathcal{O}_K)^\times$, which is a contradiction. Other cases can be shown in the same way. □

We define the sign of Δ by $\mathrm{sgn}(\Delta) = 1$ if $\Delta \equiv 1 \bmod 4$ and $\mathrm{sgn}(\Delta) = -1$ if $\Delta \equiv 3+2i \bmod 4$. For simplicity, we set

$$\varepsilon_T = \mathrm{sgn}(\Delta)\left(\frac{-1}{D_T}\right)_4^{\frac{1+\mathrm{sgn}(\Delta)}{2}} \in \{\pm 1\}.$$

Lemma 2.8. For $c \in \mathcal{C}$, we have

$$\psi_{-D_T}((4c+\Delta)) = \varepsilon_T \overline{\left(\frac{c}{D_T}\right)_4}(4c+\Delta)$$
$$\psi_{-D_T}((4c+(1+2i)\Delta)) = \varepsilon_T \overline{\left(\frac{c}{D_T}\right)_4}(4c+(1+2i)\Delta).$$

Proof. As is well-known, for an ideal $\mathfrak{a}$ of $\mathcal{O}_K$ prime to $4D_T$, it holds that

$$\psi_{-D_T}(\mathfrak{a}) = \overline{\left(\frac{-D_T}{\alpha}\right)_4}\alpha \quad (\mathfrak{a} = (\alpha),\ \alpha \equiv 1 \bmod 2+2i).$$

For example, see [Sil94, CHAPTER II, Exercise 2.34]. Since $4c+\Delta \equiv 1 \bmod 2+2i$, we have

$$\begin{aligned}\psi_{-D_T}((4c+\Delta)) &= \overline{\left(\frac{-D_T}{4c+\Delta}\right)_4}(4c+\Delta)\\ &= \left(\frac{-1}{4c+\Delta}\right)_4 \overline{\left(\frac{D_T}{4c+\Delta}\right)_4}(4c+\Delta)\\ &= \mathrm{sgn}(\Delta)\overline{\left(\frac{D_T}{4c+\Delta}\right)_4}(4c+\Delta).\end{aligned}$$

Let p_{T_i} be the number of distinct primes that divide D_{T_i} and that are congruent to $3+2i$ modulo 4. First, we consider the case of $\mathrm{sgn}(\Delta) = +1$. By the quartic reciprocity law, we can calculate as follows:

$$\begin{aligned}\left(\frac{D_T}{4c+\Delta}\right)_4 &= \prod_{i\in T_1}\left(\frac{\pi_{1,i}}{4c+\Delta}\right)_4 \prod_{j\in T_2}\left(\frac{\pi_{2,j}}{4c+\Delta}\right)_4^2 \prod_{k\in T_3}\left(\frac{\pi_{3,k}}{4c+\Delta}\right)_4^3\\ &= \prod_{i\in T_1}\left(\frac{4c+\Delta}{\pi_{1,i}}\right)_4 \prod_{j\in T_2}\left(\frac{4c+\Delta}{\pi_{2,j}}\right)_4^2 \prod_{k\in T_3}\left(\frac{4c+\Delta}{\pi_{3,k}}\right)_4^3\\ &= \prod_{i\in T_1}\left(\frac{-c}{\pi_{1,i}}\right)_4 \prod_{j\in T_2}\left(\frac{-c}{\pi_{2,j}}\right)_4^2 \prod_{k\in T_3}\left(\frac{-c}{\pi_{3,k}}\right)_4^3\\ &= (-1)^{p_{T_1}+p_{T_3}} \prod_{i\in T_1}\left(\frac{c}{\pi_{1,i}}\right)_4 \prod_{j\in T_2}\left(\frac{c}{\pi_{2,j}}\right)_4^2 \prod_{k\in T_3}\left(\frac{c}{\pi_{3,k}}\right)_4^3\\ &= \left(\frac{-1}{D_T}\right)_4\left(\frac{c}{D_T}\right)_4.\end{aligned}$$

In the same way, if $\mathrm{sgn}(\Delta) = -1$, then

$$\begin{aligned}\left(\frac{D_T}{4c+\Delta}\right)_4 &= \prod_{i\in T_1}\left(\frac{\pi_{1,i}}{4c+\Delta}\right)_4 \prod_{j\in T_2}\left(\frac{\pi_{2,j}}{4c+\Delta}\right)_4^2 \prod_{k\in T_3}\left(\frac{\pi_{3,k}}{4c+\Delta}\right)_4^3 \\ &= (-1)^{p_{T_1}+p_{T_3}} \prod_{i\in T_1}\left(\frac{4c+\Delta}{\pi_{1,i}}\right)_4 \prod_{j\in T_2}\left(\frac{4c+\Delta}{\pi_{2,j}}\right)_4^2 \prod_{k\in T_3}\left(\frac{4c+\Delta}{\pi_{3,k}}\right)_4^3 \\ &= (-1)^{p_{T_1}+p_{T_3}} \prod_{i\in T_1}\left(\frac{-c}{\pi_{1,i}}\right)_4 \prod_{j\in T_2}\left(\frac{-c}{\pi_{2,j}}\right)_4^2 \prod_{k\in T_3}\left(\frac{-c}{\pi_{3,k}}\right)_4^3 \\ &= \prod_{i\in T_1}\left(\frac{c}{\pi_{1,i}}\right)_4 \prod_{j\in T_2}\left(\frac{c}{\pi_{2,j}}\right)_4^2 \prod_{k\in T_3}\left(\frac{c}{\pi_{3,k}}\right)_4^3 \\ &= \left(\frac{c}{D_T}\right)_4.\end{aligned}$$

The rest can be proved similarly. □

Lemma 2.9. Denote the period lattice $\Omega\mathcal{O}_K$ of $E_1 : y^2 = x^3 - x$ as $\mathcal{L}_\Omega$. Let $\wp(z) = \wp(z, \mathcal{L}_\Omega)$ be the Weierstrass $\wp$-function and let $\zeta(z) = \zeta(z, \mathcal{L}_\Omega)$ be the Weierstrass ζ-function. Then for $c \in \mathcal{C}$, we have

$$\begin{aligned}&\mathcal{E}_1^*\left(\frac{c\Omega}{\Delta} + \frac{\Omega}{4}, \mathcal{L}_\Omega\right) + \mathcal{E}_1^*\left(\frac{c\Omega}{\Delta} + \frac{(1+2i)\Omega}{4}, \mathcal{L}_\Omega\right) \\ &= 2\left\{\zeta\left(\frac{c\Omega}{\Delta}\right) - \frac{\varpi}{\Omega}\overline{\left(\frac{c}{\Delta}\right)}\right\} + \frac{\wp'(c\Omega/\Delta)}{2}\left\{\frac{1}{\wp(c\Omega/\Delta) - (1+\sqrt{2})} + \frac{1}{\wp(c\Omega/\Delta) - (1-\sqrt{2})}\right\} \\ &\qquad + \sqrt{2} + \left\{\frac{2+\sqrt{2}}{\wp(c\Omega/\Delta) - (1+\sqrt{2})} - \frac{2-\sqrt{2}}{\wp(c\Omega/\Delta) - (1-\sqrt{2})}\right\},\end{aligned}$$

where $\varpi = 3.1415\ldots$ denotes pi.

Proof. For a lattice $\mathcal{L} = u\mathbb{Z} + v\mathbb{Z}$ ($\mathrm{Im}(v/u) > 0$) of $\mathbb{C}$, we set

$$s_2(\mathcal{L}) = \lim_{s\to+0} \sum_{w\in\mathcal{L}\setminus\{0\}} \frac{1}{w^2|w|^{2s}}, \quad A(\mathcal{L}) = \frac{\overline{u}v - u\overline{v}}{2\varpi i}.$$

Then, the identity $\mathcal{E}_1^*(z,\mathcal{L}) = \zeta(z,\mathcal{L}) - zs_2(\mathcal{L}) - \overline{z}A(\mathcal{L})^{-1}$ holds (cf. [GS81, Proposition 1.5]). It is easy to see $s_2(\mathcal{L}_\Omega) = 0$ and $A(\mathcal{L}_\Omega) = \Omega^2/\varpi$. Hence, we see that

$$\mathcal{E}_1^*(z,\mathcal{L}_\Omega) = \zeta(z,\mathcal{L}_\Omega) - \frac{\varpi\overline{z}}{\Omega^2}. \tag{2.1}$$

The addition formula

$$\zeta(z_1+z_2,\mathcal{L}) = \zeta(z_1,\mathcal{L}) + \zeta(z_2,\mathcal{L}) + \frac{1}{2}\frac{\wp'(z_1,\mathcal{L}) - \wp'(z_2,\mathcal{L})}{\wp(z_1,\mathcal{L}) - \wp(z_2,\mathcal{L})}$$

and equation (2.1) lead to

$$\begin{aligned}\mathcal{E}_1^*\left(\frac{c\Omega}{\Delta} + \frac{\Omega}{4}, \mathcal{L}_\Omega\right) &= \zeta\left(\frac{c\Omega}{\Delta} + \frac{\Omega}{4}\right) - \frac{\varpi}{\Omega^2}\overline{\left(\frac{c\Omega}{\Delta} + \frac{\Omega}{4}\right)} \\ &= \zeta\left(\frac{c\Omega}{\Delta}\right) + \zeta\left(\frac{\Omega}{4}\right) + \frac{1}{2}\frac{\wp'(c\Omega/\Delta) - \wp'(\Omega/4)}{\wp(c\Omega/\Delta) - \wp(\Omega/4)} - \frac{\varpi}{4\Omega} - \frac{\varpi}{\Omega}\overline{\left(\frac{c}{\Delta}\right)}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}&\mathcal{E}_1^*\left(\frac{c\Omega}{\Delta}+\frac{(1+2i)\Omega}{4},\mathcal{L}_\Omega\right)\\&=\zeta\left(\frac{c\Omega}{\Delta}\right)+\zeta\left(\frac{(1+2i)\Omega}{4}\right)+\frac{1}{2}\frac{\wp'(c\Omega/\Delta)-\wp'((1+2i)\Omega/4)}{\wp(c\Omega/\Delta)-\wp((1+2i)\Omega/4)}-\frac{(1-2i)\varpi}{4\Omega}-\frac{\varpi}{\Omega}\overline{\left(\frac{c}{\Delta}\right)}.\end{aligned}$$

Moreover from [Zha03, (2.7)], we know $\wp(\Omega/4)=1+\sqrt{2}$, $\wp'(\Omega/4)=-4-2\sqrt{2}$, $\wp((1+2i)\Omega/4)=1-\sqrt{2}$, $\wp'((1+2i)\Omega/4)=4-2\sqrt{2}$ and

$$\zeta\left(\frac{\Omega}{4}\right)+\zeta\left(\frac{(1+2i)\Omega}{4}\right)-\frac{(1-i)\varpi}{2\Omega}=\sqrt{2}.$$

By combining these results, the lemma holds. □

Theorem 2.10 (cf. [BS65]). We put $\chi=\chi(D_T)=((1+i)/D_T)_4$. Then, the following holds:

$$\begin{aligned}&\frac{\varepsilon_T\Delta}{\Omega}L_{2\Delta}(\overline{\psi_{-D_T}},1)\\&=\begin{cases}\dfrac{\sqrt{2}}{4}\displaystyle\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4+\frac{1}{\sqrt{2}}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4\frac{\wp(c\Omega/\Delta)+1}{\wp(c\Omega/\Delta)^2-2\wp(c\Omega/\Delta)-1} & ((i/D_T)_4=\pm1),\\ \dfrac{1}{8}\displaystyle\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4\left\{\frac{(1-i)\chi}{1-(1-i)\chi}\frac{\wp'(c\Omega/\Delta)}{\wp(c\Omega/\Delta)}+\frac{2\wp'(c\Omega/\Delta)(\wp(c\Omega/\Delta)-1)}{\wp(c\Omega/\Delta)^2-2\wp(c\Omega/\Delta)-1}\right\} & ((i/D_T)_4=i),\\ \dfrac{1}{4}\displaystyle\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4\frac{\wp'(c\Omega/\Delta)(\wp(c\Omega/\Delta)-1)}{\wp(c\Omega/\Delta)^2-2\wp(c\Omega/\Delta)-1} & ((i/D_T)_4=-i).\end{cases}\end{aligned}$$

Proof. Take $\Omega_T\in\mathbb{C}^\times$ so that the period lattice of the elliptic curve $E_{-D_T}: y^2=x^3+D_Tx$ is $\Omega_T\mathcal{O}_K$ and set $\alpha=\Omega/\Omega_T$. In Proposition 2.6, substituting $k=s=1,\mathfrak{g}=(4\Delta),\rho=\Omega_T/(4\Delta),\mathcal{L}=\Omega_T\mathcal{O}_K$ leads to

$$\frac{4\Delta}{\Omega_T}L_{2\Delta}(\overline{\psi_{-D_T}},1)=\sum_{\mathfrak{b}\in B}\mathcal{E}_1^*\left(\psi_{-D_T}(\mathfrak{b})\frac{\Omega_T}{4\Delta},\Omega_T\mathcal{O}_K\right).\tag{2.2}$$

Moreover, by using Lemma 2.7 and Lemma 2.8, the right-hand side of the equation (2.2) can be calculated as

$$\sum_{c\in\mathcal{C}}\mathcal{E}_1^*\left(\varepsilon_T\overline{\left(\frac{c}{D_T}\right)_4}\frac{4c+\Delta}{4\Delta}\frac{\Omega}{\alpha},\frac{\Omega}{\alpha}\mathcal{O}_K\right)+\sum_{c\in\mathcal{C}}\mathcal{E}_1^*\left(\varepsilon_T\overline{\left(\frac{c}{D_T}\right)_4}\frac{4c+(1+2i)\Delta}{4\Delta}\frac{\Omega}{\alpha},\frac{\Omega}{\alpha}\mathcal{O}_K\right).$$

Note that for $\lambda\in\mathbb{C}^\times$ and a lattice $\mathcal{L}$ of $\mathbb{C}$, $\mathcal{E}_1^*(\lambda z,\lambda\mathcal{L})=\lambda^{-1}\mathcal{E}_1^*(z,\mathcal{L})$ holds. Thus, by Lemma 2.9, we have

$$\begin{aligned}\frac{\varepsilon_T\Delta}{\Omega}L_{2\Delta}(\overline{\psi_{-D_T}},1)&=\frac{1}{4}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4\left\{\mathcal{E}_1^*\left(\frac{c\Omega}{\Delta}+\frac{\Omega}{4},\mathcal{L}_\Omega\right)+\mathcal{E}_1^*\left(\frac{c\Omega}{\Delta}+\frac{(1+2i)\Omega}{4},\mathcal{L}_\Omega\right)\right\}\\&=\frac{1}{4}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4(f_1(c)+f_2(c)+g(c)),\end{aligned}\tag{2.3}$$

where

$$f_1(c)=2\left\{\zeta\left(\frac{c\Omega}{\Delta}\right)-\frac{\varpi}{\Omega}\overline{\left(\frac{c}{\Delta}\right)}\right\},$$

$$f_2(c) = \frac{\wp'(c\Omega/\Delta)}{2}\left\{\frac{1}{\wp(c\Omega/\Delta)-(1+\sqrt{2})}+\frac{1}{\wp(c\Omega/\Delta)-(1-\sqrt{2})}\right\},$$
$$g(c) = \sqrt{2}+\left\{\frac{2+\sqrt{2}}{\wp(c\Omega/\Delta)-(1+\sqrt{2})}-\frac{2-\sqrt{2}}{\wp(c\Omega/\Delta)-(1-\sqrt{2})}\right\}.$$

The functions $f_1(c)$ and $f_2(c)$ are odd with respect to c, and $g(c)$ is even with respect to c. We prove by cases according to the value $(i/D_T)_4$.

First, we consider the case of $(i/D_T)_4 = \pm 1$. Since $(-1/D_T)_4 = 1$, the function $(c/D_T)_4$ is even with respect to c. We can take $\mathcal{C}$ so that if $c \in \mathcal{C}$, then $-c \in \mathcal{C}$ because of $(2,\Delta) = 1$. Thus $\sum_c (c/D_T)_4 f_1(c)$ and $\sum_c (c/D_T)_4 f_2(c)$ must be equal to 0. Next, we consider the case of $(i/D_T)_4 = -i$. Since $(-1/D_T)_4 = -1$, the function $(c/D_T)_4$ is odd with respect to c. As in the previous case, $\sum_c (c/D_T)_4 g(c)$ is equal to 0. Furthermore, we can take $\mathcal{C}$ so that if $c \in \mathcal{C}$, then $ic \in \mathcal{C}$. Then, the value

$$\left(\frac{c}{D_T}\right)_4\left\{\zeta\left(\frac{c\Omega}{\Delta}\right)-\frac{\varpi}{\Omega}\overline{\left(\frac{c}{\Delta}\right)}\right\}+\left(\frac{ic}{D_T}\right)_4\left\{\zeta\left(\frac{ic\Omega}{\Delta}\right)-\frac{\varpi}{\Omega}\overline{\left(\frac{ic}{\Delta}\right)}\right\}$$

is equal to 0. Hence, we have $\sum_c (c/D_T)_4 f_1(c) = 0$. Finally, we consider the case of $(i/D_T)_4 = i$. Note that the value

$$\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4\left\{\zeta\left(\frac{c\Omega}{\Delta}\right)-\frac{\varpi}{\Omega}\overline{\left(\frac{c}{\Delta}\right)}\right\}$$

does not depend on the choice of $\mathcal{C}$. In fact, we can show it by using the identities $\zeta(z+1) = \zeta(z)+\varpi$ and $\zeta(z+i) = \zeta(z)-\varpi i$. Therefore, the transformation $c \mapsto (1+i)c$ leads to

$$\begin{aligned}
&\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4\left\{\zeta\left(\frac{c\Omega}{\Delta}\right)-\frac{\varpi}{\Omega}\overline{\left(\frac{c}{\Delta}\right)}\right\}\\
&=\sum_{c\in\mathcal{C}}\left(\frac{(1+i)c}{D_T}\right)_4\left\{\zeta\left(\frac{(1+i)c\Omega}{\Delta}\right)-\frac{\varpi}{\Omega}\overline{\left(\frac{(1+i)c}{\Delta}\right)}\right\}\\
&=\chi\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4\left\{\zeta\left(\frac{c\Omega}{\Delta}\right)+\zeta\left(\frac{ic\Omega}{\Delta}\right)+\frac{1}{2}\frac{\wp'(c\Omega/\Delta)-\wp'(ic\Omega/\Delta)}{\wp(c\Omega/\Delta)-\wp(ic\Omega/\Delta)}-\frac{(1-i)\varpi}{\Omega}\overline{\left(\frac{c}{\Delta}\right)}\right\}\\
&=(1-i)\chi\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4\left\{\zeta\left(\frac{c\Omega}{\Delta}\right)+\frac{1}{4}\frac{\wp'(c\Omega/\Delta)}{\wp(c\Omega/\Delta)}-\frac{\varpi}{\Omega}\overline{\left(\frac{c}{\Delta}\right)}\right\}\\
&=(1-i)\chi\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4\left\{\zeta\left(\frac{c\Omega}{\Delta}\right)-\frac{\varpi}{\Omega}\overline{\left(\frac{c}{\Delta}\right)}\right\}+\frac{(1-i)\chi}{4}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4\frac{\wp'(c\Omega/\Delta)}{\wp(c\Omega/\Delta)}.
\end{aligned}$$

Thus, we see that

$$\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4\left\{\zeta\left(\frac{c\Omega}{\Delta}\right)-\frac{\varpi}{\Omega}\overline{\left(\frac{c}{\Delta}\right)}\right\}=\frac{(1-i)\chi}{1-(1-i)\chi}\frac{1}{4}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4\frac{\wp'(c\Omega/\Delta)}{\wp(c\Omega/\Delta)}.$$

We substitute these results into (2.3) and the theorem follows. □

We set $\mathcal{P}(c) = \wp(c\Omega/\Delta), \mathcal{P}'(c) = \wp'(c\Omega/\Delta)$ and $L^*_{2\Delta}(\overline{\psi_{-D_T}},1) = \varepsilon_T\Delta L_{2\Delta}(\overline{\psi_{-D_T}},1)$ for simplicity. Note that we have

$$v_2\left(\frac{L^*_{2\Delta}(\overline{\psi_{-D_T}},1)}{\Omega}\right)=v_2\left(\frac{L_{2\Delta}(\overline{\psi_{-D_T}},1)}{\Omega}\right).$$

As in the proof of Theorem 2.10, we take $\mathcal{C}$ so that if $c \in \mathcal{C}$, then $-c, \pm ic \in \mathcal{C}$. Let V be the subset of $\mathcal{C}$ consisting of all primary elements, that is,

$$V = \{c \in \mathcal{C} \mid c \equiv 1 \bmod 2+2i\}.$$

We can rewrite the sums over $\mathcal{C}$ in Theorem 2.10 as the sums over V. For example if $(i/D_T)_4 = 1$, then we have

$$\frac{1}{\sqrt{2}} \sum_{c \in \mathcal{C}} \left(\frac{c}{D_T}\right)_4 \frac{\mathcal{P}(c)+1}{\mathcal{P}(c)^2 - 2\mathcal{P}(c) - 1} = \sum_{c \in V} \left(\frac{c}{D_T}\right)_4 \frac{2\sqrt{2}(3\mathcal{P}(c)^2-1)}{(\mathcal{P}(c)^2 - 2\mathcal{P}(c) - 1)(\mathcal{P}(c)^2 + 2\mathcal{P}(c) - 1)}.$$

The same calculation yields the following corollary.

Corollary 2.11. Under the same conditions as Theorem 2.10, we have

$$\frac{L^*_{2\Delta}(\overline{\psi_{-D_T}}, 1)}{\Omega} = \begin{cases} \dfrac{\sqrt{2}}{4} \displaystyle\sum_{c \in \mathcal{C}} \left(\frac{c}{D_T}\right)_4 + \sum_{c \in V} \left(\frac{c}{D_T}\right)_4 \frac{2\sqrt{2}(3\mathcal{P}(c)^2-1)}{(\mathcal{P}(c)^2 - 2\mathcal{P}(c) - 1)(\mathcal{P}(c)^2 + 2\mathcal{P}(c) - 1)} & ((i/D_T)_4 = 1), \\ \dfrac{\sqrt{2}}{4} \displaystyle\sum_{c \in \mathcal{C}} \left(\frac{c}{D_T}\right)_4 + \sum_{c \in V} \left(\frac{c}{D_T}\right)_4 \frac{2\sqrt{2}\mathcal{P}(c)(\mathcal{P}(c)^2+1)}{(\mathcal{P}(c)^2 - 2\mathcal{P}(c) - 1)(\mathcal{P}(c)^2 + 2\mathcal{P}(c) - 1)} & ((i/D_T)_4 = -1), \\ \displaystyle\sum_{c \in V} \left(\frac{c}{D_T}\right)_4 \frac{\mathcal{P}'(c)}{\mathcal{P}(c)} \frac{\chi(\mathcal{P}(c)^4 - 6\mathcal{P}(c)^2 + 1) + (\mathcal{P}(c)^3 + \mathcal{P}(c))}{(\mathcal{P}(c)^2 - 2\mathcal{P}(c) - 1)(\mathcal{P}(c) + 2\mathcal{P}(c) - 1)} & ((i/D_T)_4 = i), \\ \displaystyle\sum_{c \in V} \left(\frac{c}{D_T}\right)_4 \frac{\mathcal{P}'(c)(\mathcal{P}(c)^2+1)}{(\mathcal{P}(c)^2 - 2\mathcal{P}(c) - 1)(\mathcal{P}(c)^2 + 2\mathcal{P}(c) - 1)} & ((i/D_T)_4 = -i). \end{cases}$$

Chapter 3

Zhao's method

3.1 2-adic valuation of L-value

In Corollary 2.11, we define

$$\begin{aligned}
W_1(c) &= \frac{2\sqrt{2}(3\mathcal{P}(c)^2-1)}{(\mathcal{P}(c)^2-2\mathcal{P}(c)-1)(\mathcal{P}(c)^2+2\mathcal{P}(c)-1)},\\
W_{-1}(c) &= \frac{2\sqrt{2}\mathcal{P}(c)(\mathcal{P}(c)^2+1)}{(\mathcal{P}(c)^2-2\mathcal{P}(c)-1)(\mathcal{P}(c)^2+2\mathcal{P}(c)-1)},\\
W_i(c) &= \frac{\mathcal{P}'(c)}{\mathcal{P}(c)}\frac{\chi(\mathcal{P}(c)^4-6\mathcal{P}(c)^2+1)+(\mathcal{P}(c)^3+\mathcal{P}(c))}{(\mathcal{P}(c)^2-2\mathcal{P}(c)-1)(\mathcal{P}(c)+2\mathcal{P}(c)-1)},\\
W_{-i}(c) &= \frac{\mathcal{P}'(c)(\mathcal{P}(c)^2+1)}{(\mathcal{P}(c)^2-2\mathcal{P}(c)-1)(\mathcal{P}(c)^2+2\mathcal{P}(c)-1)}.
\end{aligned}$$

Lemma 3.1. For $c \in V$, it holds that

$$v_2(W_1(c)) = v_2(W_{-1}(c)) = v_2(W_i(c)) = v_2(W_{-i}(c)) = -\frac{1}{2}.$$

Proof. [BS65, Lemma 5] shows

$$\begin{aligned}
v_2(\mathcal{P}(c)^2-2\mathcal{P}(c)-1) = v_2(\mathcal{P}(c)^2+2\mathcal{P}(c)-1) = \frac{7}{4},\\
v_2(\mathcal{P}(c)-1) = \frac{1}{2}, \quad v_2(\mathcal{P}(c)^2-1) = 1, \quad v_2(\mathcal{P}(c)^2+1) = \frac{3}{2}.
\end{aligned}$$

Thus, we have

$$v_2(3\mathcal{P}(c)^2-1) = v_2(\mathcal{P}(c)^2-3) = \frac{3}{2}, \quad v_2(\mathcal{P}(c)) = 0,$$

and $v_2(\mathcal{P}'(c)) = 3/2$ from the identity $\wp'(z)^2 = 4\wp(z)^3 - 4\wp(z)$. The claim follows from here. $\square$

We consider a summation over T of the equations in Corollary 2.11. Here, we divide the range of T into several cases and define Δ for each of these cases. We define Δ_i as the radical of D_i; that is;

$$\Delta_1 = \prod_{\pi_{1,i}\in S_1} \pi_{1,i}, \quad \Delta_2 = \prod_{\pi_{2,j}\in S_2} \pi_{2,j}, \quad \Delta_3 = \prod_{\pi_{3,k}\in S_3} \pi_{3,k}.$$

If we run T_1 over all subsets of $\{1,\dots,n\}$ and $T_2 = T_3 = \varnothing$, then we define Δ to be Δ_1. In general, we define Δ as follows:

$$\Delta = \begin{cases} \Delta_1 & (T_1 \subset \{1,\dots,n\}, T_2 = \varnothing, T_3 = \varnothing), \\ \Delta_2 & (T_1 = \varnothing, T_2 \subset \{1,\dots,m\}, T_3 = \varnothing), \\ \Delta_3 & (T_1 = \varnothing, T_2 = \varnothing, T_3 \subset \{1,\dots,\ell\}), \\ \Delta_1\Delta_2 & (T_1 \subset \{1,\dots,n\}, T_2 \subset \{1,\dots,m\}, T_3 = \varnothing), \\ \Delta_2\Delta_3 & (T_1 = \varnothing, T_2 \subset \{1,\dots,m\}, T_3 \subset \{1,\dots,\ell\}), \\ \Delta_1\Delta_3 & (T_1 \subset \{1,\dots,n\}, T_2 = \varnothing, T_3 \subset \{1,\dots,\ell\}), \\ \Delta_1\Delta_2\Delta_3 & (T_1 \subset \{1,\dots,n\}, T_2 \subset \{1,\dots,m\}, T_3 \subset \{1,\dots,\ell\}). \end{cases}$$

Here, for example, "$T_1 \subset \{1,\dots,n\}, T_2 = \varnothing, T_3 = \varnothing$" impies "$T_1$ runs over all subsets of $\{1,\dots,n\}$, T_2 and T_3 are both empty". From [ST68, Theorem 12], we see that the conductor of the elliptic curve E_{-D_T} is the square of the conductor of the Hecke character ψ_{-D_T}. Therefore by Table 2.1 and Table 2.2, it holds that the conductor of ψ_{-D_T} divides $4\Delta\mathcal{O}_K$.

Lemma 3.2. We have the following lower bound of the 2-adic valuation:

$$v_2\left(\sum_T \left(\frac{c}{D_T}\right)_4\right) \geq \begin{cases} n/2 & (T_1 \subset \{1,\dots,n\}, T_2 = \varnothing, T_3 = \varnothing), \\ m & (T_1 = \varnothing, T_2 \subset \{1,\dots,m\}, T_3 = \varnothing), \\ \ell/2 & (T_1 = \varnothing, T_2 = \varnothing, T_3 \subset \{1,\dots,\ell\}), \\ n/2+m & (T_1 \subset \{1,\dots,n\}, T_2 \subset \{1,\dots,m\}, T_3 = \varnothing), \\ m+\ell/2 & (T_1 = \varnothing, T_2 \subset \{1,\dots,m\}, T_3 \subset \{1,\dots,\ell\}), \\ (n+\ell)/2 & (T_1 \subset \{1,\dots,n\}, T_2 = \varnothing, T_3 \subset \{1,\dots,\ell\}), \\ n/2+m+\ell/2 & (T_1 \subset \{1,\dots,n\}, T_2 \subset \{1,\dots,m\}, T_3 \subset \{1,\dots,\ell\}). \end{cases}$$

Proof. We only prove the case of $D_T = D_{T_1}$. We only need to show that

$$\sum_{T_1 \subset \{1,\dots,n\}} \left(\frac{c}{D_{T_1}}\right)_4 = \left\{1 + \left(\frac{c}{\pi_{1,1}}\right)_4\right\} \cdots \left\{1 + \left(\frac{c}{\pi_{1,n}}\right)_4\right\}.$$

We show by induction on n. Clearly, it holds for $n = 1$. Suppose it is true for $1,\dots,n-1$. Then, we have

$$\begin{aligned} \sum_{T_1 \subset \{1,\dots,n\}} \left(\frac{c}{D_{T_1}}\right)_4 &= \sum_{T_1 \subset \{1,\dots,n-1\}} \left(\frac{c}{D_{T_1}}\right)_4 + \sum_{\substack{T_1 \subset \{1,\dots,n\} \\ n \in T_1}} \left(\frac{c}{D_{T_1}}\right)_4 \\ &= \sum_{T_1 \subset \{1,\dots,n-1\}} \left(\frac{c}{D_{T_1}}\right)_4 + \left(\frac{c}{\pi_{1,n}}\right)_4 \sum_{T_1 \subset \{1,\dots,n-1\}} \left(\frac{c}{D_{T_1}}\right)_4 \\ &= \left\{1 + \left(\frac{c}{\pi_{1,n}}\right)_4\right\} \sum_{T_1 \subset \{1,\dots,n-1\}} \left(\frac{c}{D_{T_1}}\right)_4 \\ &= \left\{1 + \left(\frac{c}{\pi_{1,1}}\right)_4\right\} \cdots \left\{1 + \left(\frac{c}{\pi_{1,n}}\right)_4\right\}, \end{aligned}$$

where the last equality follows from the induction hypothesis. Thus, it is true for n. This completes the proof. □

Proposition 3.3. The following holds:

$$v_2\left(\sum_T \frac{L^*_{2\Delta}(\overline{\psi_{-D_T}},1)}{\Omega}\right) \geq \begin{cases} \dfrac{n-1}{2} & (T_1 \subset \{1,\dots,n\}, T_2=\varnothing, T_3=\varnothing),\\ \dfrac{2m-1}{2} & (T_1=\varnothing, T_2\subset\{1,\dots,m\}, T_3=\varnothing),\\ \dfrac{\ell-1}{2} & (T_1=\varnothing, T_2=\varnothing, T_3\subset\{1,\dots,\ell\}),\\ \dfrac{n+2m-1}{2} & (T_1\subset\{1,\dots,n\}, T_2\subset\{1,\dots,m\}, T_3=\varnothing),\\ \dfrac{2m+\ell-1}{2} & (T_1=\varnothing, T_2\subset\{1,\dots,m\}, T_3\subset\{1,\dots,\ell\}),\\ \dfrac{n+\ell-1}{2} & (T_1\subset\{1,\dots,n\}, T_2=\varnothing, T_3\subset\{1,\dots,\ell\}),\\ \dfrac{n+2m+\ell-1}{2} & (T_1\subset\{1,\dots,n\}, T_2\subset\{1,\dots,m\}, T_3\subset\{1,\dots,\ell\}). \end{cases}$$

Proof. We only prove the case of $D_T = D_{T_1}$. Consider the summation over T_1 for the equations in Corollary 2.11. Then, we have $\Delta = \Delta_1$ and

$$\sum_{T_1} \frac{L^*_{2\Delta_1}(\overline{\psi_{-D_{T_1}}},1)}{\Omega} = \begin{cases} \dfrac{\sqrt{2}}{4}\displaystyle\sum_{T_1}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_{T_1}}\right)_4 + \sum_{c\in V} W_1(c)\sum_{T_1}\left(\frac{c}{D_{T_1}}\right)_4 & ((i/D_{T_1})_4 = 1),\\ \dfrac{\sqrt{2}}{4}\displaystyle\sum_{T_1}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_{T_1}}\right)_4 + \sum_{c\in V} W_{-1}(c)\sum_{T_1}\left(\frac{c}{D_{T_1}}\right)_4 & ((i/D_{T_1})_4 = -1),\\ \displaystyle\sum_{c\in V} W_i(c)\sum_{T_1}\left(\frac{c}{D_{T_1}}\right)_4 & ((i/D_{T_1})_4 = i),\\ \displaystyle\sum_{c\in V} W_{-i}(c)\sum_{T_1}\left(\frac{c}{D_{T_1}}\right)_4 & ((i/D_{T_1})_4 = -i), \end{cases}$$

where T_1 runs over all subsets of $\{1,\dots,n\}$. By Lemma 3.1 and Lemma 3.2, for any $\circ \in \{\pm1, \pm i\}$, we have

$$v_2\left(\sum_{c\in V} W_\circ(c)\sum_{T_1}\left(\frac{c}{D_{T_1}}\right)_4\right) \geq \min_{c\in V}\left\{v_2(W_\circ(c)) + v_2\left(\sum_{T_1}\left(\frac{c}{D_{T_1}}\right)_4\right)\right\} = -\frac{1}{2} + \frac{n}{2}.$$

Since

$$\sum_{c\in\mathcal{C}}\left(\frac{c}{D_{T_1}}\right)_4 = \begin{cases} 0 & (T_1 \neq \varnothing),\\ \#\mathcal{C} & (T_1 = \varnothing), \end{cases}$$

it holds that

$$v_2\left(\frac{\sqrt{2}}{4}\sum_{T_1}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_{T_1}}\right)_4\right) = v_2(\#\mathcal{C}) - \frac{3}{2} \geq 2n - \frac{3}{2} > \frac{n-1}{2}.$$

The proposition follows from this. □

3.2 Proof of the main theorems

Theorem 3.4. Let ψ_{-D} be the Hecke character associated to the elliptic curve $E_{-D} : y^2 = x^3 + Dx$ over K and $\Omega = \int_1^\infty dx/\sqrt{x^3 - x}$ a period of $E_1 : y^2 = x^3 - x$. Then, we have

$$v_2\left(\frac{L_2(\overline{\psi_{-D}}, 1)}{\Omega}\right) \geq \begin{cases} \dfrac{n-2}{2} & (D = D_1^{(n)}), \\ \dfrac{2m-3}{2} & (D = D_2^{(m)}), \\ \dfrac{\ell-2}{2} & (D = D_3^{(\ell)}). \end{cases}$$

Proof. We only prove the case of $D = D_1^{(n)}$. When $T_1 = \{1, \ldots, n\}$, we see that $E_{-D_{T_1}} = E_{-D_1^{(n)}}$ and $L^*_{2\Delta_1}(\overline{\psi_{-D_{T_1}}}, 1) = L^*_2(\overline{\psi_{-D_1^{(n)}}}, 1)$ holds by Corollary 2.5. When $T_1 = \varnothing$, the elliptic curve $E_{-D_{T_1}} = E_{-1}$ has bad reduction at the prime $(1+i)\mathcal{O}_K$. Therefore, we have

$$L^*_{2\Delta_1}(\overline{\psi_{-1}}, 1) = L^*_{\Delta_1}(\overline{\psi_{-1}}, 1) = L^*(\overline{\psi_{-1}}, 1) \prod_{i=1}^{n} \left(1 - \frac{\overline{\psi_{-1}}((\pi_{1,i}))}{N(\pi_{1,i})}\right).$$

Since $L(\overline{\psi_{-1}}, 1) = \Omega/(2\sqrt{2})$ (*cf.* [BS65, p.87]), we obtain

$$v_2\left(\frac{L^*_{2\Delta_1}(\overline{\psi_{-1}}, 1)}{\Omega}\right) = \sum_{i=1}^{n} v_2\left(\pi_{1,i} - \left(\frac{-1}{\pi_{1,i}}\right)_4\right) - \frac{3}{2} \geq n - \frac{3}{2}. \tag{3.1}$$

We prove the theorem by induction on n. For $n = 1$, by Proposition 3.3, we see that the 2-adic valuation of

$$\frac{L^*_{2\Delta_1}(\overline{\psi_{-1}}, 1)}{\Omega} + \frac{L^*_2(\overline{\psi_{-D_1^{(n)}}}, 1)}{\Omega} \tag{3.2}$$

is greater than $-1/2$. Since the 2-adic valuation of the first term in (3.2) is greater than or equal to $-1/2$ by (3.1), the valuation of the second term must also be greater than or equal to $-1/2$. Thus, it holds for $n = 1$. Suppose it is true for $1, \ldots, n-1$. Then by Proposition 3.3, the 2-adic valuation of

$$\frac{L^*_{2\Delta_1}(\overline{\psi_{-1}}, 1)}{\Omega} + \sum_{\varnothing \neq T_1 \subsetneq \{1,\ldots,n\}} \frac{L^*_{2\Delta_1}(\overline{\psi_{-D_{T_1}}}, 1)}{\Omega} + \frac{L^*_2(\overline{\psi_{-D_1^{(n)}}}, 1)}{\Omega} \tag{3.3}$$

is greater than $(n-2)/2$. The valuation of the first term in (3.3) is greater than or equal to $(n-2)/2$ by (3.1). By using the induction hypothesis, it holds that

$$\begin{aligned}
& v_2\left(\sum_{\varnothing \neq T_1 \subsetneq \{1,\ldots,n\}} \frac{L^*_{2\Delta_1}(\overline{\psi_{-D_{T_1}}}, 1)}{\Omega}\right) \\
&= v_2\left(\sum_{\varnothing \neq T_1 \subsetneq \{1,\ldots,n\}} \frac{L^*_2(\overline{\psi_{-D_{T_1}}}, 1)}{\Omega} \prod_{\pi_{1,i} \nmid D_{T_1}} \left(1 - \frac{\overline{\psi_{-D_{T_1}}}((\pi_{1,i}))}{N(\pi_{1,i})}\right)\right) \\
&\geq \min_{\varnothing \neq T_1 \subsetneq \{1,\ldots,n\}} \left\{\frac{\#T_1 - 2}{2} + \sum_{\pi_{1,i} \nmid D_{T_1}} v_2\left(\pi_{1,i} - \overline{\psi_{-D_{T_1}}}((\pi_{1,i}))\right)\right\} \\
&\geq \min_{\varnothing \neq T_1 \subsetneq \{1,\ldots,n\}} \left\{\frac{\#T_1 - 2}{2} + \frac{n - \#T_1}{2}\right\}
\end{aligned}$$

$$= \frac{n-2}{2}.$$

Thus, it also holds for n and we obtain the theorem. □

Theorem 3.5. Under the same conditions as Theorem 3.4, we have

$$v_2\left(\frac{L_2(\overline{\psi_{-D}},1)}{\Omega}\right) \geq \begin{cases} \frac{n+m-2}{2} & (D = D_1^{(n)}D_2^{(m)}), \\ \frac{m+\ell-2}{2} & (D = D_2^{(m)}D_3^{(\ell)}), \\ \frac{n+\ell-2}{2} & (D = D_1^{(n)}D_3^{(\ell)}), \\ \frac{n+m+\ell-2}{2} & (D = D_1^{(n)}D_2^{(m)}D_3^{(\ell)}). \end{cases}$$

Proof. We only prove the case of $D = D_1^{(n)}D_2^{(m)}$ by double induction on n and m based on the following steps (see Figure 3.1 and Figure 3.2).

Step 1 It holds for $(1, m)$ for all m.

Step 2 It holds for $(n, 1)$ for all n.

Step 3 If it holds for $(n_0, m_0) \neq (n, m)$ $(1 \leq n_0 \leq n, 1 \leq m_0 \leq m)$, then (n, m) holds.

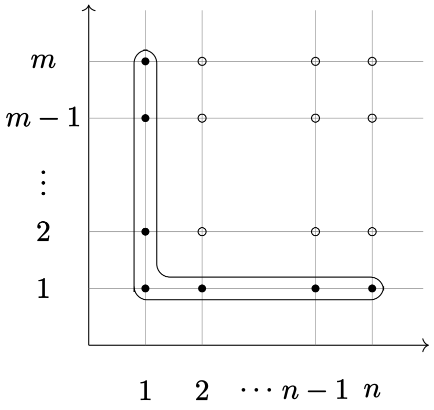

Figure 3.1: Step 1 and Step 2

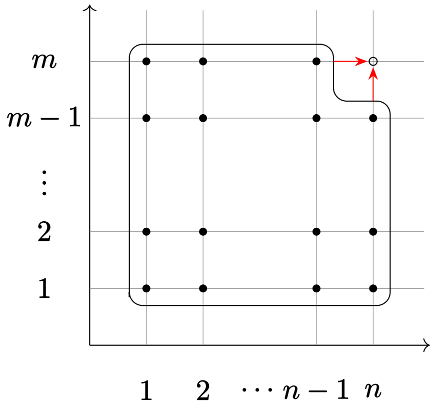

Figure 3.2: Step 3

First, we show Step 1 by induction on m. For $m = 1$, the 2-adic valuation of

$$\underbrace{\frac{L^*_{2\Delta_1\Delta_2}(\overline{\psi_{-1}},1)}{\Omega}}_{T_1=T_2=\varnothing} + \underbrace{\frac{L^*_{2\Delta_1\Delta_2}(\overline{\psi_{-D_2^{(1)}}},1)}{\Omega}}_{T_1=\varnothing,T_2=\{1\}} + \underbrace{\frac{L^*_{2\Delta_1\Delta_2}(\overline{\psi_{-D_1^{(1)}}},1)}{\Omega}}_{T_1=\{1\},T_2=\varnothing} + \underbrace{\frac{L^*_2(\overline{\psi_{-D_1^{(1)}D_2^{(1)}}},1)}{\Omega}}_{T_1=\{1\},T_2=\{1\}} \tag{3.4}$$

is greater than 0 from Proposition 3.3. Therefore, we need to show the first three terms of (3.4) is greater than or equal to 0. For the first term, we see that

$$\begin{aligned} v_2\left(\frac{L^*_{2\Delta_1\Delta_2}(\overline{\psi_{-1}},1)}{\Omega}\right) &= v_2\left(\frac{L^*_2(\overline{\psi_{-1}},1)}{\Omega}\left(1 - \frac{\overline{\psi_{-1}}((\pi_{1,1}))}{N(\pi_{1,1})}\right)\left(1 - \frac{\overline{\psi_{-1}}((\pi_{2,1}))}{N(\pi_{2,1})}\right)\right) \\ &\geq -\frac{3}{2} + 1 + 1 \\ &> 0. \end{aligned}$$

For the second term, by Theorem 3.4, we have

$$v_2\left(\frac{L^*_{2\Delta_1\Delta_2}(\overline{\psi_{-D_2^{(1)}}},1)}{\Omega}\right)=v_2\left(\frac{L^*_2(\overline{\psi_{-D_2^{(1)}}},1)}{\Omega}\left(1-\frac{\overline{\psi_{-D_2^{(1)}}}((\pi_{1,1}))}{N(\pi_{1,1})}\right)\right)\geq -\frac{1}{2}+1>0.$$

For the third term, we can show that the 2-adic valuation is greater than 0 similarly to the second term. Thus it holds for $m=1$. Suppose it is true for $1,\dots,m-1$. Then the 2-adic valuation of

$$\begin{aligned}&\underbrace{\frac{L^*_{2\Delta_1\Delta_2}(\overline{\psi_{-1}},1)}{\Omega}}_{T_1=T_2=\varnothing}+\sum_{\varnothing\neq T_2\subset\{1,\dots,m\}}\frac{L^*_{2\Delta_1\Delta_2}(\overline{\psi_{-D_{T_2}}},1)}{\Omega}+\underbrace{\frac{L^*_{2\Delta_1\Delta_2}(\overline{\psi_{-D_1^{(1)}}},1)}{\Omega}}_{T_1=\{1\},T_2=\varnothing}\\&+\sum_{\varnothing\neq T_2\subsetneq\{1,\dots,m\}}\frac{L^*_{2\Delta_1\Delta_2}(\overline{\psi_{-D_1^{(1)}D_{T_2}}},1)}{\Omega}+\underbrace{\frac{L^*_2(\overline{\psi_{-D_1^{(1)}D_2^{(m)}}},1)}{\Omega}}_{T_1=\{1\},T_2=\{1,\dots,m\}}\end{aligned}\tag{3.5}$$

is greater than $(m-1)/2$ from Proposition 3.3. Therefore, we need to show the first four terms of (3.5) is greater than or equal to $(m-1)/2$. For the first term, we see that

$$\begin{aligned}v_2\left(\frac{L^*_{2\Delta_1\Delta_2}(\overline{\psi_{-1}},1)}{\Omega}\right)&=v_2\left(\frac{L^*_2(\overline{\psi_{-1}},1)}{\Omega}\left(1-\frac{\overline{\psi_{-1}}((\pi_{1,1}))}{N(\pi_{1,1})}\right)\prod_{j=1}^{m}\left(1-\frac{\overline{\psi_{-1}}((\pi_{2,j}))}{N(\pi_{2,j})}\right)\right)\\&\geq-\frac{3}{2}+1+m\\&>\frac{m-1}{2}.\end{aligned}$$

For the second term, by Theorem 3.4, we have

$$\begin{aligned}&v_2\left(\sum_{\varnothing\neq T_2\subset\{1,\dots,m\}}\frac{L^*_{2\Delta_1\Delta_2}(\overline{\psi_{-D_{T_2}}},1)}{\Omega}\right)\\&\geq\min_{\varnothing\neq T_2\subset\{1,\dots,m\}}\left\{v_2\left(\frac{L^*_2(\overline{\psi_{-D_{T_2}}},1)}{\Omega}\left(1-\frac{\overline{\psi_{-D_{T_2}}}((\pi_{1,1}))}{N(\pi_{1,1})}\right)\prod_{\pi_{2,j}\nmid D_{T_2}}\left(1-\frac{\overline{\psi_{-D_{T_2}}}((\pi_{2,j}))}{N(\pi_{2,j})}\right)\right)\right\}\\&\geq\min_{\varnothing\neq T_2\subset\{1,\dots,m\}}\left\{\frac{2\#T_2-3}{2}+1+(m-\#T_2)\right\}\\&>\frac{m-1}{2}.\end{aligned}$$

For the third term, by Theorem 3.4, it follows

$$\begin{aligned}v_2\left(\frac{L^*_{2\Delta_1\Delta_2}(\overline{\psi_{-D_1^{(1)}}},1)}{\Omega}\right)&=v_2\left(\frac{L^*_2(\overline{\psi_{-D_1^{(1)}}},1)}{\Omega}\prod_{j=1}^{m}\left(1-\frac{\overline{\psi_{-D_1^{(1)}}}((\pi_{2,j}))}{N(\pi_{2,j})}\right)\right)\\&\geq-\frac{1}{2}+\frac{1}{2}\cdot m\\&=\frac{m-1}{2}.\end{aligned}$$

For the fourth term, by the induction hypothesis, it holds

$$v_2\left(\sum_{\varnothing\neq T_2\subsetneq\{1,\dots,m\}}\frac{L^*_{2\Delta_1\Delta_2}(\overline{\psi_{-D_1^{(1)}D_{T_2}}},1)}{\Omega}\right)$$

$$\begin{aligned}
&\geq \min_{\varnothing\neq T_2\subsetneq\{1,\ldots,m\}}\left\{v_2\left(\frac{L_2^*(\overline{\psi_{-D_1^{(1)}D_{T_2}}},1)}{\Omega}\prod_{\pi_{2,j}\nmid D_{T_2}}\left(1-\frac{\overline{\psi_{-D_1^{(1)}D_{T_2}}}((\pi_{2,j}))}{N(\pi_{2,j})}\right)\right)\right\}\\
&\geq \frac{1+\#T_2-2}{2}+\frac{1}{2}\cdot(m-\#T_2)\\
&=\frac{m-1}{2}.
\end{aligned}$$

Thus it holds for m and Step 1 is done.

By a similar calculation, Step 2 can be shown by induction on n. We show Step 3. Suppose it is true for (n_0,m_0) $(1\leq n_0\leq n, 1\leq m_0\leq m, (n_0,m_0)\neq(n,m))$. The 2-adic valuation of

$$\begin{aligned}
&\underbrace{\frac{L_{2\Delta_1\Delta_2}^*(\overline{\psi_{-1}},1)}{\Omega}}_{T_1=T_2=\varnothing}+\sum_{\varnothing\neq T_2\subset\{1,\ldots,m\}}\frac{L_{2\Delta_1\Delta_2}^*(\overline{\psi_{-D_{T_2}}},1)}{\Omega}+\sum_{\varnothing\neq T_1\subset\{1,\ldots,n\}}\frac{L_{2\Delta_1\Delta_2}^*(\overline{\psi_{-D_{T_1}}},1)}{\Omega}\\
&+\sum_{\substack{\varnothing\neq T_1\subsetneq\{1,\ldots,n\}\\ \varnothing\neq T_2\subsetneq\{1,\ldots,m\}}}\frac{L_{2\Delta_1\Delta_2}^*(\overline{\psi_{-D_{T_1}D_{T_2}}},1)}{\Omega}+\sum_{\varnothing\neq T_1\subsetneq\{1,\ldots,n\}}\frac{L_{2\Delta_1\Delta_2}^*(\overline{\psi_{-D_{T_1}D_2^{(m)}}},1)}{\Omega}\\
&+\sum_{\varnothing\neq T_2\subsetneq\{1,\ldots,m\}}\frac{L_{2\Delta_1\Delta_2}^*(\overline{\psi_{-D_1^{(n)}D_{T_2}}},1)}{\Omega}+\underbrace{\frac{L_2^*(\overline{\psi_{-D_1^{(n)}D_2^{(m)}}},1)}{\Omega}}_{T_1=\{1,\ldots,n\},T_2=\{1,\ldots,m\}}
\end{aligned}\tag{3.6}$$

is greater than $(n+m-2)/2$ from Proposition 3.3 (see Figure 3.3).

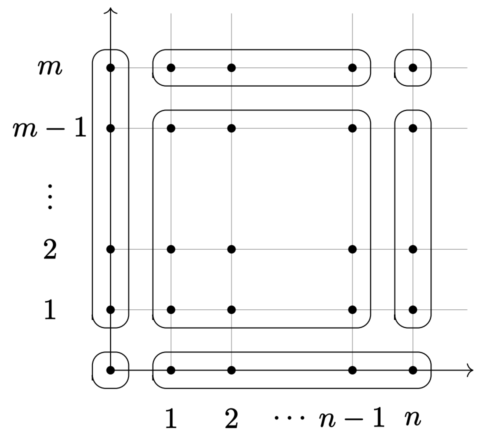

Figure 3.3: Equation (3.6)

Therefore, we need to show the first sixth terms of (3.6) is greater than or equal to $(n+m-2)/2$. We calculate the 2-adic valuation for the first term, second term and fourth term. For the others term, one could calculate similarly. For the first term, we see that

$$\begin{aligned}
v_2\left(\frac{L_{2\Delta_1\Delta_2}^*(\overline{\psi_{-1}},1)}{\Omega}\right)&=v_2\left(\frac{L_2^*(\overline{\psi_{-1}},1)}{\Omega}\prod_{i=1}^{n}\left(1-\frac{\overline{\psi_{-1}}((\pi_{1,i}))}{N(\pi_{1,i})}\right)\prod_{j=1}^{m}\left(1-\frac{\overline{\psi_{-1}}((\pi_{2,j}))}{N(\pi_{2,j})}\right)\right)\\
&\geq -\frac{3}{2}+n+m\\
&>\frac{n+m-2}{2}.
\end{aligned}$$

For the second term, by Theorem 3.4, it follows

$$
\begin{aligned}
&v_2\left(\sum_{\varnothing\neq T_2\subset\{1,\dots,m\}}\frac{L^*_{2\Delta_1\Delta_2}(\overline{\psi_{-D_{T_2}}},1)}{\Omega}\right)\\
&\geq \min_{\varnothing\neq T_2\subset\{1,\dots,m\}}\left\{v_2\left(\frac{L^*_2(\overline{\psi_{-D_{T_2}}},1)}{\Omega}\prod_{i=1}^{n}\left(1-\frac{\overline{\psi_{-D_{T_2}}}((\pi_{1,i}))}{N(\pi_{1,i})}\right)\prod_{\pi_{2,j}\nmid D_{T_2}}\left(1-\frac{\overline{\psi_{-D_{T_2}}}((\pi_{2,j}))}{N(\pi_{2,j})}\right)\right)\right\}\\
&\geq \min_{\varnothing\neq T_2\subset\{1,\dots,m\}}\left\{\frac{2\#T_2-3}{2}+1\cdot n+1\cdot(m-\#T_2)\right\}\\
&> \frac{n+m-2}{2}.
\end{aligned}
$$

For the fourth term, by the induction hypothesis, it holds

$$
\begin{aligned}
&v_2\left(\sum_{\substack{\varnothing\neq T_1\subsetneq\{1,\dots,n\}\\ \varnothing\neq T_2\subsetneq\{1,\dots,m\}}}\frac{L^*_{2\Delta_1\Delta_2}(\overline{\psi_{-D_{T_1}D_{T_2}}},1)}{\Omega}\right)\\
&\geq \min_{\substack{\varnothing\neq T_1\subsetneq\{1,\dots,n\}\\ \varnothing\neq T_2\subsetneq\{1,\dots,m\}}}\left\{v_2\left(\frac{L^*_2(\overline{\psi_{-D_{T_1}D_{T_2}}},1)}{\Omega}\prod_{\pi_{1,i}\nmid D_{T_1}}\left(1-\frac{\overline{\psi_{-D_{T_1}D_{T_2}}}((\pi_{1,i}))}{N(\pi_{1,i})}\right)\prod_{\pi_{2,j}\nmid D_{T_2}}\left(1-\frac{\overline{\psi_{-D_{T_1}D_{T_2}}}((\pi_{2,j}))}{N(\pi_{2,j})}\right)\right)\right\}\\
&\geq \min_{\substack{\varnothing\neq T_1\subsetneq\{1,\dots,n\}\\ \varnothing\neq T_2\subsetneq\{1,\dots,m\}}}\left\{\frac{\#T_1+\#T_2-2}{2}+\frac{1}{2}\cdot(n-\#T_1)+\frac{1}{2}\cdot(m-\#T_2)\right\}\\
&= \frac{n+m-2}{2}.
\end{aligned}
$$

Thus it is true for $(n_0, m_0) = (n, m)$ and Step 3 is done. This completes the proof. □

3.3 Numerical Examples

As mentioned in Remark 1.8, the lower bounds in Theorem 3.4 and Theorem 3.5 are expected to be sharp in the sense that there exist elliptic curves E_{-D} for which equality holds. We have listed the 2-adic valuation for the case $D = D_1^{(1)}$ and $D = D_1^{(1)}D_2^{(1)}$. Here, we have arranged it in ascending order of the absolute value of D.

$v_2(L_2(\overline{\psi_{-D}}, 1)/\Omega)$	D	$(i/D)_4$	$v_2(L_2(\overline{\psi_{-D}}, 1)/\Omega)$	D	$(i/D)_4$
$-1/2$	$2i-1$	i	$-1/2$	$-6i+19$	$-i$
0	-3	-1	1	$-20i+1$	1
$-1/2$	$-2i+3$	$-i$	0	$20i-3$	-1
∞	$-4i+1$	1	$-1/2$	$-14i+15$	i
$-1/2$	$2i-5$	$-i$	∞	$-12i+17$	1
$-1/2$	$6i-1$	i	∞	$20i-7$	1
0	$-4i+5$	-1	0	$-4i+21$	-1
∞	-7	1	$-1/2$	$10i+19$	$-i$
$-1/2$	$-2i+7$	i	$-1/2$	$22i-5$	$-i$
$-1/2$	$-6i-5$	$-i$	0	$-20i-11$	-1
0	$8i-3$	-1	∞	-23	1
0	$8i+5$	-1	$-1/2$	$10i-21$	$-i$
1	$-4i+9$	1	$-1/2$	$-14i+19$	$-i$
$-1/2$	$10i-1$	i	0	$20i+13$	-1
$-1/2$	$10i+3$	$-i$	∞	$-24i+1$	1
∞	$-8i-7$	1	∞	$-8i-23$	1
0	-11	-1	0	$-24i+5$	-1
0	$-4i-11$	-1	$-1/2$	$-18i-17$	i
$-1/2$	$-10i+7$	i	0	$-16i-19$	-1
$-1/2$	$6i+11$	$-i$	1	$-4i+25$	1
$-1/2$	$2i-13$	$-i$	$-1/2$	$-22i-13$	$-i$
$-1/2$	$-10i-9$	i	$-1/2$	$6i-25$	i
∞	$-12i-7$	1	∞	$-12i-23$	1
$-1/2$	$14i-1$	i	$-1/2$	$26i-1$	i
$-1/2$	$-2i+15$	i	$-1/2$	$-26i-5$	$-i$
0	$8i+13$	-1	$-1/2$	$-22i+15$	i
1	$-4i-15$	1	$-1/2$	$-2i+27$	$-i$
∞	$-16i+1$	1	$-1/2$	$26i-9$	i
$-1/2$	$-10i-13$	$-i$	0	$-20i-19$	-1
$-1/2$	$-14i-9$	i	∞	$-12i+25$	1
0	$16i+5$	-1	$-1/2$	$-22i-17$	i
$-1/2$	$2i-17$	i	$-1/2$	$26i+11$	$-i$
0	$-12i+13$	-1	0	$28i+5$	-1
$-1/2$	$14i+11$	$-i$	$-1/2$	$-14i-25$	i
1	$16i+9$	1	$-1/2$	$-10i+27$	$-i$
$-1/2$	$-18i-5$	$-i$	$-1/2$	$18i+23$	i
∞	$-8i+17$	1	0	$-4i+29$	-1
0	-19	-1	$-1/2$	$-6i-29$	$-i$
$-1/2$	$18i+7$	i	1	$16i+25$	1
$-1/2$	$10i-17$	i	2	$20i-23$	1

Table 3.1: 2-adic valuation for $D = D_1^{(1)}$

$v_2(L_2(\overline{\psi_{-D}},1)/\Omega)$	$D_1^{(1)}$	$D_2^{(1)}$	$(i/D)_4$
∞	-3	$(2i-1)^2$	1
0	$-2i+3$	$(2i-1)^2$	i
0	$2i-1$	$(-3)^2$	i
1	$-4i+1$	$(2i-1)^2$	-1
$1/2$	$2i-5$	$(2i-1)^2$	i
0	$2i-1$	$(-2i+3)^2$	$-i$
0	$6i-1$	$(2i-1)^2$	$-i$
∞	$-4i+5$	$(2i-1)^2$	1
$1/2$	$-2i+3$	$(-3)^2$	$-i$
$1/2$	-7	$(2i-1)^2$	-1
∞	$-4i+1$	$(-3)^2$	1
$1/2$	$2i-1$	$(-4i+1)^2$	i
1	-3	$(-2i+3)^2$	1
0	$2i-5$	$(-3)^2$	$-i$
$1/2$	-3	$(-4i+1)^2$	-1
$1/2$	$-4i+1$	$(-2i+3)^2$	-1
∞	$6i-1$	$(-3)^2$	i
1	-11	$(2i-1)^2$	1
$1/2$	$-4i+5$	$(-3)^2$	-1
0	$-2i+3$	$(-4i+1)^2$	$-i$
2	-7	$(-3)^2$	1
∞	$2i-1$	$(2i-5)^2$	$-i$
0	$2i-5$	$(-2i+3)^2$	i
0	$6i-1$	$(-2i+3)^2$	$-i$
0	$2i-1$	$(6i-1)^2$	$-i$
1	$-4i+5$	$(-2i+3)^2$	1
$3/2$	-3	$(2i-5)^2$	1
$1/2$	-7	$(-2i+3)^2$	-1
∞	$2i-5$	$(-4i+1)^2$	$-i$
0	$2i-1$	$(-4i+5)^2$	i
1	-19	$(2i-1)^2$	1
∞	-11	$(-3)^2$	-1
$1/2$	$6i-1$	$(-4i+1)^2$	i
0	$-2i+3$	$(2i-5)^2$	i
∞	$-4i+5$	$(-4i+1)^2$	-1
0	$2i-1$	$(-7)^2$	i
1	-3	$(6i-1)^2$	1
$1/2$	-23	$(2i-1)^2$	-1
$3/2$	-7	$(-4i+1)^2$	1
∞	$-4i+1$	$(2i-5)^2$	-1

$v_2(L_2(\overline{\psi_{-D}},1)/\Omega)$	$D_1^{(1)}$	$D_2^{(1)}$	$(i/D)_4$
$1/2$	-3	$(-4i+5)^2$	-1
0	$-2i+3$	$(6i-1)^2$	i
∞	-11	$(-2i+3)^2$	1
∞	-3	$(-7)^2$	-1
$1/2$	$-2i+3$	$(-4i+5)^2$	$-i$
1	$-4i+1$	$(6i-1)^2$	-1
1	-31	$(2i-1)^2$	-1
2	$-4i+1$	$(-4i+5)^2$	1
∞	-19	$(-3)^2$	-1
0	$6i-1$	$(2i-5)^2$	$-i$
0	$-2i+3$	$(-7)^2$	$-i$
1	$-4i+5$	$(2i-5)^2$	1
$1/2$	-11	$(-4i+1)^2$	-1
0	$2i-5$	$(6i-1)^2$	i
$5/2$	$-4i+1$	$(-7)^2$	1
1	-7	$(2i-5)^2$	-1
∞	-23	$(-3)^2$	1
∞	-43	$(2i-1)^2$	1
∞	$2i-5$	$(-4i+5)^2$	$-i$
$1/2$	-47	$(2i-1)^2$	-1
$3/2$	$-4i+5$	$(6i-1)^2$	1
$3/2$	-19	$(-2i+3)^2$	1
0	$6i-1$	$(-4i+5)^2$	i
1	-7	$(6i-1)^2$	-1
$1/2$	$2i-5$	$(-7)^2$	$-i$
$1/2$	$2i-1$	$(-11)^2$	i
2	-31	$(-3)^2$	1
$3/2$	-7	$(-4i+5)^2$	1
$1/2$	$6i-1$	$(-7)^2$	i
2	-23	$(-2i+3)^2$	-1
$1/2$	$-4i+5$	$(-7)^2$	-1
$3/2$	-11	$(2i-5)^2$	1
1	-19	$(-4i+1)^2$	-1
∞	-3	$(-11)^2$	-1
∞	-3	$(-11)^2$	-1
∞	-43	$(-3)^2$	-1
$3/2$	-23	$(-4i+1)^2$	1
$1/2$	-31	$(-2i+3)^2$	-1
1	-11	$(6i-1)^2$	1
3	-47	$(-3)^2$	1

Table 3.2: 2-adic valuation for $D = D_1^{(1)} D_2^{(1)}$

Part II

Recurrence formula

Chapter 4

Motivation

Which prime number p can be written as the sum of two cubes of rational numbers? This is one of the classical Diophantine problems and there are various works (*cf.* [DV18], [Yin22]). This problem is related to the existence of $\mathbb{Q}$-rational points of the curve $A_p : x^3 + y^3 = p$. The curve A_p has the structure of an elliptic curve defined over $\mathbb{Q}$ with the point $\infty = [1 : -1 : 0]$. For an odd prime number p, we see that $A_p(\mathbb{Q})_{\mathrm{tors}} = \{\infty\}$. Therefore an odd prime number p is written as the sum of two cubes if and only if the rank of A_p over $\mathbb{Q}$ is not 0. [Sat86] shows the upper bound

$$\operatorname{rank} A_p(\mathbb{Q}) \leq \begin{cases} 0 & (p \equiv 2, 5 \bmod 9), \\ 1 & (p \equiv 4, 7, 8 \bmod 9), \\ 2 & (p \equiv 1 \bmod 9). \end{cases}$$

In addition to the above upper bound, we explain that it is possible to determine whether the rank of $A_p(\mathbb{Q})$ is even or odd.

For an elliptic curve E defined over a number field K, let us denote the p^n-Selmer group by $\mathrm{Sel}_{p^n}(E/K)$ and p^∞-Selmer group by

$$\mathrm{Sel}_{p^\infty}(E/K) := \varinjlim_n \mathrm{Sel}_{p^n}(E/K).$$

The p^∞-Selmer group $\mathrm{Sel}_{p^\infty}(E/K)$ is a cofinitely generated $\mathbb{Z}_p$-module (*cf.* [Gre99]) and sits in the exact sequence

$$0 \longrightarrow E(K) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathrm{Sel}_{p^\infty}(E/K) \longrightarrow Ш(E/K)[p^\infty] \longrightarrow 0.$$

Therefore if the Tate–Shafarevich group $Ш(E/K)$ is finite, the rank of $E(K)$ over $\mathbb{Q}$ is equal to the corank of $\mathrm{Sel}_{p^\infty}(E/K)$ over $\mathbb{Z}_p$. The following theorem is called the *p-parity conjecture* that is proved by Nekovář [Nek09].

Theorem 4.1 ([Nek09, Theorem 1])**.** Let k be a totally real number field, k_0/k a finite abelian extension and k'/k_0 a Galois extension of odd degree. Let E be an elliptic curve over k; assume that at least one of the following conditions is satisfied:

(i) E is modular (over k) and $2 \nmid [k : \mathbb{Q}]$;

(ii) $j(E) \notin \mathcal{O}_k$;

then, for each prime number $p \neq 2$, the parity conjecture

$$\operatorname{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E/k') \equiv \operatorname{ord}_{s=1} L(E/k', s) \mod 2$$

holds. If $k = \mathbb{Q}$, then the statement also holds for $p = 2$.

Let $\varepsilon(A_p/\mathbb{Q})$ be the sign of the functional equation for the Hasse–Weil L-function $L(A_p/\mathbb{Q}, s)$ of A_p. By [ZK87, Table 1], the sign $\varepsilon(A_p/\mathbb{Q})$ is computed as $+1$ if p is congruent to $1, 2, 5$ modulo 9 and -1 otherwise. Hence if $\text{Ш}(A_p/\mathbb{Q})$ is finite, we have

$$(-1)^{\operatorname{rank} A_p(\mathbb{Q})} = \varepsilon(A_p/\mathbb{Q}) = \begin{cases} +1 & (p \equiv 1, 2, 5 \bmod 9), \\ -1 & (\text{otherwise}). \end{cases}$$

from Theorem 4.1. Thus for the case where $p \equiv 1 \bmod 9$ (resp. $p \equiv 4, 7, 8 \bmod 9$), the rank of A_p is 0 or 2 (resp. 1).

The remaining problem is essentially whether the rank of A_p is 0 or 2 for the case where p is congruent to 1 modulo 9. In the paper [RZ95], Rodríguez-Villegas and Zagier have given three necessary and sufficient conditions that the rank is equal to 2 under the Birch and Swinnerton–Dyer (BSD) conjecture. One of the conditions is described in terms of a recurrence formula although they did not give the details of the proof.

In this thesis, we give a similar formula for the elliptic curve $E_{-p} : y^2 = x^3 + px$. A 2-descent [Sil86, Proposition 6.2] shows the upper bound

$$\operatorname{rank} E_{-p}(\mathbb{Q}) \leq \begin{cases} 0 & (p \equiv 7, 11 \bmod 16), \\ 1 & (p \equiv 3, 5, 13, 15 \bmod 16), \\ 2 & (p \equiv 1, 9 \bmod 16). \end{cases}$$

For the case where p is congruent to $1, 9$ modulo 16, the sign of functional equation of the Hasse–Weil L-function of E_{-p} over $\mathbb{Q}$ is $+1$. Similarly for the case of A_p, we see that $\operatorname{rank} E_{-p}(\mathbb{Q}) = 0$ or 2 if we assume the Tate–Shafarevich group is finite. We obtain the following result.

Theorem 4.2 ([Nom22b, Theorem 1.1])**.** Let p be a prime number which is congruent to $1, 9$ modulo 16. If the rank of E_{-p} over $\mathbb{Q}$ is equal to 2, then p divides $f_{3(p-1)/8}(0)$, where the polynomial $f_n(t) \in \mathbb{Z}[t]$ is defined by the recurrence formula

$$f_{n+1}(t) = -12(t+1)(t+2)f_n'(t) + (4n+1)(2t+3)f_n(t) - 2n(2n-1)(t^2+3t+3)f_{n-1}(t).$$

The initial condition is $f_0(t) = 1$, $f_1(t) = 2t + 3$. Moreover if we assume the BSD conjecture, then the converse is also true.

This theorem tells us a criterion for determining whether the rank is 2 or not although we may not be able to decide the rank exactly when we use a descent algorithm. In fact, using `RankBounds` command of Magma [BCP97], we can see that the exact rank of $E_{-12553}(\mathbb{Q})$ is not determined. In addition, there is an advantage that the recurrence formula in Theorem 4.2 can be implemented in the same way by everyone in the same environment without advanced functions. As a reference, we summarize a behavior of $\{f_n(t)\}_{n\geq 1}$ in Table 4.3 and Table 4.4.

Now, we return to the elliptic curve $A_p : x^3 + y^3 = p$. We tried to recover the proof of Theorem 4.3 below. Although we could not obtain the proof of Theorem 4.3, we obtain Theorem 4.4 instead. Our recurrence formula (4.2) is simpler than (4.1). In Table 4.1 and Table 4.2, we show the first several terms for the two recurrence formulas. The degree of the polynomial and the number of terms of (4.2) are less than (4.1). Moreover, the time of calculating the percentage of rank 2 up to $p < 5000$ on Magma [BCP97] version V2.24-5 on dual-core Intel Core i5 processor (3.1 GHz), 8GM RAM and mac OS Catalina, the formula (4.2) is about 34 seconds faster than (4.1). (The percentage of rank 2 is about 37% up to $p < 5000$.) Perhaps we may make the recurrence formula (4.2) simpler. A procedure to obtain the recurrence formula (4.2) is essentially the same as [RZ95].

Theorem 4.3 ([RZ95, Theorem 3]). Let p be a prime number which is congruent to 1 modulo 9, the rank of A_p over $\mathbb{Q}$ is equal to 2, then p divides $a_{(p-1)/3}(0)$, where the polynomial $a_n(t) \in \mathbb{Z}[t]$ is defined by the recurrence formula

$$a_{n+1}(t) = -(1-8t^3)a_n'(t) - (16n+3)t^2 a_n(t) - 4n(2n-1)t a_{n-1}(t). \tag{4.1}$$

The initial condition is $a_0(t) = 1$, $a_1(t) = -3t^2$. Moreover if we assume the BSD conjecture, then the converse is also true.

Theorem 4.4 ([Nom22b, Theorem 1.3]). Let p be a prime number which is congruent to 1 modulo 9, the rank of A_p over $\mathbb{Q}$ is equal to 2, then p divides $x_{(p-1)/3}(0)$, where the polynomial $x_n(t) \in \mathbb{Z}[t]$ is defined by the recurrence formula

$$x_{n+1}(t) = -2(1-8t^3)x_n'(t) - 8nt^2 x_n(t) - n(2n-1)t x_{n-1}(t). \tag{4.2}$$

The initial condition is $x_0(t) = 1$, $x_1(t) = 0$. Moreover if we assume the BSD conjecture, then the converse is also true.

We now explain the proof of Theorem 4.2. For the case where p is congruent to $1, 9$ modulo 16, we see that rank $E_{-p}(\mathbb{Q}) = 2$ if and only if $L(E_{-p}/\mathbb{Q}, 1) = 0$ under the BSD conjecture. The calculation $L(E_{-p}/\mathbb{Q}, 1)$ reduces to $L(\psi^{2k-1}, k)$ for some Hecke character ψ and some positive integer k. More precisely, by a theory of p-adic L-functions, there exists a mod p congruence relation between the algebraic part of $L(E_{-p}/\mathbb{Q}, 1)$ and that of $L(\psi^{2k-1}, k)$. Therefore with the estimate $|L(E_{-p}/\mathbb{Q}, 1)|$, it holds that $L(E_{-p}/\mathbb{Q}, 1) = 0$ if and only if p divides the algebraic part $L_{E,k}$ of $L(\psi^{2k-1}, k)$, that is, the p-adic valuation of $L_{E,k}$ is positive. We write the algebraic part of $L(\psi^{2k-1}, k)$ in terms of a recurrence formula by using the method of [RZ93].

Part II is organized as follows. In Chapter 5, we show the rank of E_{-p} is equal to 2 if and only if p divides the algebraic part of $L(\psi^{2k-1}, k)$. In Chapter 6, we represent the special value $L(\psi^{2k-1}, k)$ as some special value of the derivative by the Maass–Shimura operator ∂_k of some modular form. In Chapter 7, we write the special value of ∂_k-derivative of the modular form as the constant term of some polynomial that is defined by a recurrence formula.

n	$a_n(t)$
0	1
1	$-3t^2$
2	$9t^4 + 2t$
3	$-27t^6 - 18t^3 - 2$
4	$81t^8 + 108t^5 + 36t^2$
5	$-243t^{10} - 540t^7 - 360t^4 + 152t$
6	$729t^{12} + 2430t^9 + 2700t^6 - 16440t^3 - 152$
7	$-2187t^{14} + 10206t^{11} - 17010t^8 + 1311840t^5 + 24240t^2$
8	$6561t^{16} + 40824t^{13} + 95256t^{10} - 99234720t^7 - 2974800t^4 + 6848t$
9	$-19683t^{18} - 157464t^{15} - 489888t^{12} + 7449816240t^9 + 359465040t^6 - 578304t^3 - 6848$

Table 4.1: the first 10 polynomials for $a_n(t)$

n	$x_n(t)$
0	1
1	0
2	$-t$
3	2
4	$-33t^2$
5	$76t$
6	$-339t^3$
7	$4314t^2$
8	$-72687t^4 - 3424t$
9	$228168t^3 + 6848$

Table 4.2: the first 10 polynomials for $x_n(t)$

p	$p \mid f_{3(p-1)/8}(0)$	p	$p \mid f_{3(p-1)/8}(0)$
17	false	257	false
41	false	281	true
73	true	313	false
89	true	337	true
97	false	353	true
113	true	401	false
137	false	409	false
193	false	433	false
233	true	449	false
241	false	457	false

Table 4.3: the constant term $f_{3(p-1)/8}(0)$

n	$f_n(t)$
0	1
1	$2t + 3$
2	$-6t^2 - 18t - 9$
3	$12t^3 + 54t^2 + 108t + 81$
4	$60t^4 + 360t^3 + 1296t^2 + 2268t + 1377$
5	$-1512t^5 - 11340t^4 - \cdots - 34992t^2 - 13122t + 2187$
6	$21816t^6 + 196344t^5 + \cdots + 1027890t^2 + 433026t + 80919$

Table 4.4: the first 7 polynomials for $f_n(t)$

Chapter 5

Congruence relation between the algebraic parts

Here we show that there exists a mod p congruence relation between the algebraic part of $L(E_{-p}/\mathbb{Q}, 1)$ and that of some special value of a Hecke L-function associated to the elliptic curve $E_{-1} : y^2 = x^3 + x$. In the rest of part II, let $\varpi = 3.1415\ldots$ denote pi.

5.1 Interpolation formula of a p-adic L-function

In this section, we state an interpolation formula of a p-adic L-function that interpolates special values of Hecke L-functions associated to elliptic curves with complex multiplication and good ordinary reduction at p. Such p-adic L-functions have been studied by, for example, Manin–Vishik [VM74] and Katz [Kat76]. We refer to the de Shalit's book [Sha87] for the contents of this section.

Let K be an imaginary quadratic field of discriminant $-d_K$ and F/K an extension of a field. Fix $\overline{\mathbb{Q}}$ as an algebraic closure of $\mathbb{Q}$. We write a Hecke character of F whose image belongs to $\overline{\mathbb{Q}}$ by χ and its conductor by $\mathfrak{f}$. For an integral ideal $\mathfrak{m}$, $L_{\mathfrak{m}}(\chi, s)$ denotes the Hecke L-function $L(\chi, s)$ of χ omitting all Euler factors corresponding to the primes that divide $\mathfrak{m}$. It is well known that $L(\chi, s)$ admits an analytic continuation on $\mathbb{C}$ if $\chi \neq 1$ and satisfies a certain functional equation (For example, see [Tat67], [Iwa19]). When $F = K$, the Hecke character χ is said to be of type (k, j) if $\chi(\alpha\mathcal{O}_K) = \alpha^k\overline{\alpha}^j$ with $\alpha \equiv 1 \bmod \mathfrak{f}$. Fix embeddings $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Denote $[-, K(\mathfrak{f}p^\infty)/K]$ by the Artin map for global class field theory associated to the modulus $\mathfrak{f}p^\infty$. The Hecke character χ can be extended continuously to the Galois character

$$\widetilde{\chi} : \mathrm{Gal}(K(\mathfrak{f}p^\infty)/K) \to \mathbb{C}_p^\times, \quad \widetilde{\chi}([\mathfrak{a}, K(\mathfrak{f}p^\infty)/K]) = \chi(\mathfrak{a})$$

via the embedding i_p (*cf.* [Wei56]).

We assume p splits as $\mathfrak{p}\overline{\mathfrak{p}}$ in K and the embedding $i_p : \overline{\mathbb{Q}_p} \hookrightarrow \mathbb{C}_p$ is compatible with $\mathfrak{p}$-adic topology. Let $F' = K(\mathfrak{f}\overline{\mathfrak{p}}^\infty)$ and $F_n = K(\mathfrak{f}\mathfrak{p}^n)$ so that $F'F_n = K(\mathfrak{f}\mathfrak{p}^n\overline{\mathfrak{p}}^\infty)$. For an integral ideal $\mathfrak{g}$ of K and a Hecke character ε of type (k, j) whose conductor dividing $\mathfrak{g}p^\infty$, We write $\varepsilon = \varphi^k\overline{\varphi}^j\chi$, where φ is a Hecke character of conductor prime to $\mathfrak{p}$ and type $(1, 0)$, and χ is a finite character. Set

$$S = \{\gamma \in \mathrm{Gal}(F'F_n/K) \mid \gamma|_{F'} = [\mathfrak{p}^n, F'/K]\},$$

where n is the exact power of $\mathfrak{p}$ dividing the conductor of ε.

Definition 5.1. We define the Gauss sum for ε by

$$G(\varepsilon) = \frac{\varphi^k\overline{\varphi}^j(\mathfrak{p}^n)}{p^n}\sum_{\gamma\in S}\chi(\gamma)(\zeta_n^\gamma)^{-1}.$$

Remark 5.2. The Gauss sum $G(\varepsilon)$ is independent of the decomposition $\varepsilon = \varphi^k\overline{\varphi}^j\chi$.

Theorem 5.3 (*cf.* [Sha87, Theorem 4.12])**.** The following hold:

(i) Let $\mathfrak{f}$ be any non-trivial integral ideal of K, and $\mathfrak{p}$ a split prime $(\mathfrak{p}, f) = 1$. Then there exist periods $\Omega \in \mathbb{C}^\times$ and $\Omega_p \in \mathbb{C}_p^\times$, and a unique p-adic integral measure $\mu(\mathfrak{f})$ on $\mathcal{G}(\mathfrak{f}) = \mathrm{Gal}(K(\mathfrak{f}\mathfrak{p}^\infty)/K)$, such that for any Hecke character ε of conductor dividing $\mathfrak{f}\mathfrak{p}^\infty$ and type $(k, 0)$, $k \geq 1$,

$$\Omega_p^{-k}\int_{\mathcal{G}(\mathfrak{f})}\widetilde{\varepsilon}(\sigma)d\mu(\mathfrak{f};\sigma) = \Omega^{-k}\frac{(k-1)!}{(2\varpi)^k}G(\varepsilon)\left(1 - \frac{\varepsilon(\mathfrak{p})}{p}\right)\cdot L_{\mathfrak{f}}(\varepsilon^{-1}, 0).$$

(ii) If $\mathfrak{f} \mid \mathfrak{g}$ and $\overline{\mu}(\mathfrak{g})$ is the measure induced from $\mu(\mathfrak{g})$ on $\mathcal{G}(f)$, then

$$\overline{\mu}(\mathfrak{g}) = \prod(1 - [\mathfrak{l}, K(\mathfrak{f}\mathfrak{p}^\infty)/K]^{-1})\cdot\mu(\mathfrak{f}),$$

where the product is over all $\mathfrak{l}$ dividing $\mathfrak{g}$ but not $\mathfrak{f}$.

Remark 5.4. As stated in [Sha87, REMARKS (i), p.76], the claim (i) of Theorem 5.3 holds if $\mathfrak{f}$ is replaced by $\mathfrak{f}\mathfrak{g}^\infty$ with $(\mathfrak{f}\mathfrak{g}, \mathfrak{p}) = 1$ from the claim (ii) of Theorem 5.3.

Let ζ_n be the primitive p^n root of unity fixed as [Sha87, p.79, CONVENTION]. Also, let $(\Omega, \Omega_p) \in (\mathbb{C}^\times \times \mathbb{C}_p^\times)/\overline{\mathbb{Q}}^\times$ be the pair of complex period and p-adic period as in [Sha87, p.68, DEFINITION].

Theorem 5.5 (*cf.* [Sha87, Theorem 4.14])**.** Let $\mathfrak{g}$ be an integral ideal of K, and p a split rational prime, $(p, \mathfrak{g}) = 1$. Let μ be the measure $\mu(\mathfrak{g}\overline{\mathfrak{p}}^\infty)$ on $\mathcal{G} = \mathrm{Gal}(K(\mathfrak{g}p^\infty)/K)$ (see Theorem 5.3 and Remark 5.4). Then the following formula, both sides of which lie in $\overline{\mathbb{Q}}$, holds for any Hecke character ε of conductor dividing $\mathfrak{g}\mathfrak{p}^\infty$, and of type (k, j), $0 \leq -j < k$:

$$\Omega_p^{j-k}\int_{\mathcal{G}}\widetilde{\varepsilon}(\sigma)d\mu(\sigma) = \Omega^{j-k}\frac{(k-1)!}{(2\varpi)^k}\left(\frac{\sqrt{d_K}}{2\varpi}\right)^j G(\varepsilon)\left(1 - \frac{\varepsilon(\mathfrak{p})}{p}\right)\cdot L_{\mathfrak{g}\overline{\mathfrak{p}}}(\varepsilon^{-1}, 0).$$

5.2 Congruence relation

Let E_{-p} be the elliptic curve $y^2 = x^3 + px$ defined over $\mathbb{Q}$. Suppose p satisfies $p \equiv 1, 9 \bmod 16$ and splits as $\mathfrak{p}\overline{\mathfrak{p}}$ in the ring of integers $\mathcal{O}_K$ of $K = \mathbb{Q}(i)$. If necessary by replacing $\overline{\mathfrak{p}}$ by $\mathfrak{p}$, we may assume a generator $\pi = a + bi$ of $\mathfrak{p}$ satisfies

$$a \equiv 1 \bmod 4, \quad b \equiv -\left(\frac{p-1}{2}\right)!a \bmod p.$$

We fix embeddings $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ so that i_p is compatible with $\mathfrak{p}$-adic topology. Let $\Omega_E = \Gamma(1/4)^2/(2\varpi^{1/2})$ be the real period of $E_{-1} : y^2 = x^3 + x$. We define the algebraic part of $L(E_{-p}/\mathbb{Q}, 1)$ to be

$$S_p = \frac{2p^{1/4}L(E_{-p}/\mathbb{Q}, 1)}{\Omega_E}.$$

The algebraic part S_p is a rational integer [BS65, Theorem 1]. The BSD conjecture predicts that S_p is equal to the order of the Tate–Shafarevich group if rank $E_{-p}(\mathbb{Q}) = 0$ and is 0 otherwise.

Proposition 5.6. The algebraic part S_p is 0 if and only if S_p is congruent to 0 modulo p.

Proof. For an elliptic curve E defined over $\mathbb{Q}$ of conductor N, [RZ95, Proposition 2] shows

$$|L(E/\mathbb{Q},1)| < (4N)^{1/4}\left(\log\frac{\sqrt{N}}{8\varpi}+\gamma\right)+c_0,$$

where $\gamma = 0.577\cdots$ is Euler's constant and $c_0 = \zeta(1/2)^2 = 2.13263\cdots$. Since $p \equiv 1 \bmod 4$, we see that the conductor of E_{-p} is $64p^2$ and obtain $|S_p| < p$. The claim follows from this. $\square$

The elliptic curve $E_{-1} : y^2 = x^3 + x$ has complex multiplication by $\mathcal{O}_K$. Let ψ be the Hecke character of K associated to E_{-1} and let χ be the quartic character such that $L(E_{-p}/\mathbb{Q}, s) = L(\psi\chi, s)$. These characters are explicitly given by

$$\psi(\mathfrak{a}) = \left(\frac{-1}{\alpha}\right)_4 \alpha = (-1)^{(a-1)/2}\alpha \quad \text{if } (\mathfrak{a},4)=1,$$
$$\chi(\mathfrak{a}) = \overline{\left(\frac{\alpha}{p}\right)_4} \quad \text{if } (\mathfrak{a},p)=1,$$

where $\alpha = a + bi$ is the primary generator of $\mathfrak{a}$ and $(\cdot/\cdot)_4$ is the quartic residue character (*cf.* [Sil94, CHAPTER II, Exercice 2.34]). Let k be a positive interger. We define the algebraic part of $L(\psi^{2k-1}, k)$ to be

$$L_{E,k} = \frac{2^{k+1}3^{k-1}\varpi^{k-1}(k-1)!}{\Omega_E^{2k-1}} L(\psi^{2k-1}, k).$$

Lemma 5.7. Let p be a prime number such that $p \equiv 1, 9 \bmod 16$ and $k = (3p+1)/4$. For all non-zero integral ideals $\mathfrak{a}$ of $\mathcal{O}_K$ which is prime to $4p$, we have

$$\chi(\mathfrak{a}) \equiv \left(\frac{\alpha}{\overline{\alpha}}\right)^{k-1} \bmod p,$$

where α is the primary generator of $\mathfrak{a}$.

Proof. Since $3(N(\pi)-1) = 4(k-1)$, we have

$$\alpha^{k-1} \equiv \left(\frac{\alpha^3}{\pi}\right)_4 \bmod \pi, \quad \alpha^{k-1} \equiv \left(\frac{\alpha^3}{\overline{\pi}}\right)_4 \bmod \overline{\pi}.$$

We take $a \in \mathfrak{p}, b \in \overline{\mathfrak{p}}$ so that $a + b = 1$. Then by the Chinese Remainder Theorem, we have

$$\alpha^{k-1} \equiv a\left(\frac{\alpha^3}{\overline{\pi}}\right)_4 + b\left(\frac{\alpha^3}{\pi}\right)_4 \bmod p\mathcal{O}_K, \tag{5.1}$$
$$\overline{\alpha}^{k-1} \equiv a\left(\frac{\overline{\alpha}^3}{\overline{\pi}}\right)_4 + b\left(\frac{\overline{\alpha}^3}{\pi}\right)_4 \bmod p\mathcal{O}_K. \tag{5.2}$$

Since the equation (5.1) multiplied by $(\overline{\alpha}^3/\pi)_4$ equals to the equation (5.2) multiplied by $(\alpha^3/\pi)_4$, it holds that

$$\left(\frac{\overline{\alpha}^3}{\pi}\right)_4 \alpha^{k-1} \equiv \left(\frac{\alpha^3}{\pi}\right)_4 \overline{\alpha}^{k-1} \bmod p\mathcal{O}_K.$$

Therefore we obtain

$$\frac{\alpha^{k-1}}{\overline{\alpha}^{k-1}} \equiv \left(\frac{\alpha^3}{\pi}\right)_4 \left(\frac{\alpha^3}{\overline{\pi}}\right)_4 = \left(\frac{\alpha}{p}\right)^3 = \chi(\mathfrak{a}) \bmod p\mathcal{O}_K.$$

$\square$

Proposition 5.8. Under the same assumptions as in Lemma 5.7, we have the following mod p congruence relation:

$$\overline{\pi} S_p \equiv u 2^{4k-5} 3^{3k-3} L_{E,k} \mod p$$

for some $u \in \mathcal{O}_K^\times$.

Proof. It is straightforward to check

$$L(\psi\chi, 1) = \sum_{(\mathfrak{a},4p)=1} \chi(\mathfrak{a}) \frac{1}{\overline{\psi}(\mathfrak{a}) N\mathfrak{a}^s}\Bigg|_{s=0},$$

$$L(\psi^{2k-1}, k) = \sum_{(\mathfrak{a},4)=1} \left(\frac{\alpha}{\overline{\alpha}}\right)^{k-1} \frac{1}{\overline{\psi}(\mathfrak{a}) N\mathfrak{a}^s}\Bigg|_{s=0}.$$

We set $\varepsilon_1(\mathfrak{a}) = \chi(\mathfrak{a})\psi(\mathfrak{a}), \varepsilon_2(\mathfrak{a}) = (\psi(\mathfrak{a})/\overline{\psi}(\mathfrak{a}))^{k-1}\psi(\mathfrak{a})$ so that $L_{4p}(\varepsilon_1^{-1}, 0) = L(\psi\chi, 1)$ and $L_4(\varepsilon_2^{-1}, 0) = L(\psi^{2k-1}, k)$. Since p splits in K (or the elliptic curve E_{-1} is ordinary at p), by Theorem 5.5, the following identities, both sides of which lie in $\overline{\mathbb{Q}}$, holds:

$$\frac{1}{\Omega_p} \int_{\mathcal{G}} \widetilde{\varepsilon_1}(\sigma) d\mu(\sigma) = \frac{1}{\Omega} G(\varepsilon_1) L_{4p}(\varepsilon_1^{-1}, 0),$$

$$\frac{1}{\Omega_p^{2k-1}} \int_{\mathcal{G}} \widetilde{\varepsilon_2}(\sigma) d\mu(\sigma) = \frac{(k-1)!}{\Omega^{2k-1}} \varpi^{k-1} G(\varepsilon_2) \left(1 - \frac{\varepsilon_2(\mathfrak{p})}{p}\right)^2 L_4(\varepsilon_2^{-1}, 0),$$

where μ is the p-adic measure on $\mathcal{G} = \mathrm{Gal}(K(4p^\infty)/K)$. Lemma 5.7 shows

$$\left| \int_{\mathcal{G}} \widetilde{\varepsilon_1}(\sigma) d\mu(\sigma) - \int_{\mathcal{G}} \widetilde{\varepsilon_2}(\sigma) d\mu(\sigma) \right|_\pi \leq \max_{(\mathfrak{a},4p)=1} |\varepsilon_1(\mathfrak{a}) - \varepsilon_2(\mathfrak{a})|_\pi \leq \frac{1}{p}.$$

Therefore we obtain the congruence relation

$$\frac{\Omega_p}{\Omega} G(\varepsilon_1) L_{4p}(\varepsilon_1^{-1}, 0) \equiv \frac{\Omega_p^{2k-1}(k-1)!}{\Omega^{2k-1}} \varpi^{k-1} L_4(\varepsilon_2^{-1}, 0) \bmod p.$$

By [Sha87, p.91, Lemma] and [Lox77, p.8, (14)], $G(\varepsilon_1)^2$ is equal to $\sqrt{p\pi}$ up to units in $\mathcal{O}_K^\times$ and $G(\varepsilon_2)$ is equal to 1. Moreover, [Sha87, p.9-10] shows $\Omega_p^{p-1} \equiv \overline{\pi}^{-1} \bmod p$. Hence it follows that

$$\overline{\pi} S_p \equiv u 2^{4k-5} 3^{3k-3} L_{E,k} \bmod p \tag{5.3}$$

for some $u \in \mathcal{O}_K^\times$. □

Remark 5.9. It is known that $(\frac{p-1}{2})!^2 \equiv -1 \bmod p$ and [Lem00, Corollary 6.6] shows

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv \pi + \overline{\pi} \mod p.$$

Thus (5.3) can be rewritten as

$$S_p \equiv \pm \left(\frac{p-1}{4}\right)!^2 2^{4k-5} 3^{3k-3} L_{E,k} \mod p.$$

The proof of Proposition 5.8 essentially shows Rodríguez-Villegas' and Zagier's congruence relation [RZ95, p.7]

$$S_{A,p} \equiv (-3)^{(p-10)/3} \left(\frac{p-1}{3}\right)!^2 L_{A,k} \mod p,$$

where $S_{A,p}$ is the algebraic part of the special value $L(A_p/\mathbb{Q}, 1)$. The algebraic number $L_{A,k}$ is explained in detail below.

By Corollary 5.10, we only need to calculate the algebraic part $L_{E,k}$. Actually, $L_{E,k}$ is the square of a rational integer. We calculate the square root of it in Chapter 6.

Let ψ' be the Hecke character of $\mathbb{Q}(\sqrt{-3})$ associated to $A_1 : x^3 + y^3 = 1$. We define the algebraic part of $L(\psi'^{2k-1}, k)$ to be

$$L_{A,k} = 3\nu \left(\frac{2\varpi}{3\sqrt{3}\Omega_A^2} \right)^{k-1} \frac{(k-1)!}{\Omega_A} L(\psi'^{2k-1}, k),$$

where $\Omega_A = \Gamma(1/3)^3/(2\varpi\sqrt{3})$ is the real period of A_1 and $\nu = 2$ if $k \equiv 2 \bmod 6$, $\nu = 1$ otherwise. For the case where p is congruent to 1 modulo 9, we see that the rank of A_p is equal to 0 if and only if p divides $L_{A,k}$ in the same way for E_{-p}.

Corollary 5.10. If the rank of E_{-p} (resp. A_p) is equal to 2, then p divides the algebraic part $L_{E,k}$ (resp. the algebraic part $L_{A,k}$). Moreover, if we assume the BSD conjecture, then the converse is true.

Proof. It follows from Coates–Wiles theorem [CW77], Proposition 5.6 and Proposition 5.8. □

Chapter 6

Square formula of L-value

6.1 Maass–Shimura operator

Unless otherwise stated, we denote by $\Gamma \subset SL_2(\mathbb{R})$ a congruence subgroup. Let $M_k(\Gamma)$ be the space of holomorphic modular forms of weight k for Γ. In general, $M_k^*(\Gamma)$ denotes the space of differentiable modular form, possibly with some character or multiplier system. For a function f on $\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ with values in $\mathbb{C} \cup \{\infty\}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we define the usual slash operator $\cdot|[\gamma]_k$ by

$$(f|[\gamma]_k)(z) := (cz+d)^{-k} f(\gamma z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

Let D be the differential operator

$$D = \frac{1}{2\varpi i}\frac{d}{dz} = q\frac{d}{dq} \quad (q = e^{2\varpi i z}).$$

By a simple calculation, we see that

$$(Df)\left(\frac{az+b}{cz+d}\right) = (cz+d)^{k+2}(Df)(z) + \frac{k}{2\varpi i}c(cz+d)^{k+1}f(z). \tag{6.1}$$

Therefore the operator D does not preserve modularity. On the other hand, the Maass–Shimura operator

$$\partial_k = D - \frac{k}{4\varpi y} \quad (z = x + iy)$$

preserves it although does not preserve holomorphy. We define $\partial_k^{(h)}$ by $\partial_{k+2h-2} \circ \partial_{k+2h-4} \circ \cdots \circ \partial_{k+2} \circ \partial_k$.

Proposition 6.1. The Maass–Shimura operator is compatible with the slash operator, that is, for $\gamma \in \Gamma$, we have

$$\partial_k(f|[\gamma]_k) = (\partial_k f)|[\gamma]_{k+2}.$$

In particular, if $f \in M_k^*(\Gamma)$, then we have $\partial_k^{(h)} f \in M_{k+2h}^*(\Gamma)$.

Proof. It follows from the equation (6.1). □

Proposition 6.2. The following holds:

$$\partial_k^{(h)} = \sum_{j=0}^{h} \binom{h}{j} \frac{(h+k-1)!}{(j+k-1)!} \left(\frac{-1}{4\varpi y}\right)^{h-j} D^j.$$

Proof. It can be easily shown by using induction on h. □

Proposition 6.3 ([RZ93, p.4, (16)]). The following holds:

$$\partial_k^{(h)}\left(\frac{1}{(mz+n)^k}\right) = \frac{(h+k-1)!}{(k-1)!}\left(\frac{-1}{4\varpi y}\frac{m\bar{z}+n}{mz+n}\right)^h \frac{1}{(mz+n)^k}.$$

Proof. By Proposition 6.2, we calculate as follows:

$$\begin{aligned}
\partial_k^{(h)}\left(\frac{1}{(mz+n)^k}\right) &= \sum_{j=0}^{h}\binom{h}{j}\frac{(h+k-1)!}{(j+k-1)!}\left(\frac{-1}{4\varpi y}\right)^{h-j} D^j \frac{1}{(mz+n)^k} \\
&= \sum_{j=0}^{h}\binom{h}{j}\frac{(h+k-1)!}{(j+k-1)!}\left(\frac{-1}{4\varpi y}\right)^{h-j} \frac{1}{(2\varpi i)^j}\frac{(-m)^j k(k+1)\cdots(k+j-1)}{(mz+n)^{k+j}} \\
&= \frac{(h+k-1)!}{(k-1)!}\frac{1}{(mz+n)^k}\sum_{j=0}^{h}\binom{h}{j}\left(\frac{-m}{2\varpi i(mz+n)}\right)^j\left(\frac{-1}{4\varpi y}\right)^{h-j} \\
&= \frac{(h+k-1)!}{(k-1)!}\frac{1}{(mz+n)^k}\left(\frac{-m}{2\varpi i(mz+n)}+\frac{-1}{4\varpi y}\right)^h \\
&= \frac{(h+k-1)!}{(k-1)!}\left(\frac{-1}{4\varpi y}\frac{m\bar{z}+n}{mz+n}\right)^h\frac{1}{(mz+n)^k}.
\end{aligned}$$

□

We define the h-th generalized Laguerre polynomial to be

$$L_h^{\alpha}(z) = \sum_{j=0}^{\infty}\binom{h+\alpha}{h-j}\frac{(-z)^j}{j!} \quad (h\in\mathbb{Z}_{\geq 0},\ \alpha\in\mathbb{C}).$$

In the special case $\alpha = 1/2,\ -1/2$, we see that

$$H_{2n}(z) = (-4)^n n! L_n^{-1/2}(z^2), \quad H_{2n+1}(z) = 2(-4)^n n! z L_n^{1/2}(z^2), \tag{6.2}$$

where

$$H_n(z) = \sum_{0\leq j\leq n/2}\frac{n!}{j!(n-2j)!}(-1)^j(2z)^{n-2j}$$

is the n-th Hermite polynomial.

Proposition 6.4 ([RZ93, p.3, (9)]). The following holds:

$$\partial_k^{(h)}\left(\sum_{n=0}^{\infty}a(n)e^{2\varpi inz}\right) = \frac{(-1)^h h!}{(4\varpi y)^h}\sum_{n=0}^{\infty}a(n)L_h^{k-1}(4\varpi ny)e^{2\varpi inz}.$$

In particular for $k = 1/2, 3/2$, we have

$$\begin{aligned}
\partial_{1/2}^{(h)}\left(\sum_{n=0}^{\infty}a(n)e^{\varpi in^2 z}\right) &= \frac{(-1)^h h!}{(4\varpi y)^h}\sum_{n=0}^{\infty}a(n)L_h^{-1/2}(2n^2\varpi y)e^{\varpi in^2 z}, \\
\partial_{3/2}^{(h)}\left(\sum_{n=0}^{\infty}a(n)e^{\varpi in^2 z}\right) &= \frac{(-1)^h h!}{(4\varpi y)^h}\sum_{n=0}^{\infty}a(n)L_h^{1/2}(2n^2\varpi y)e^{\varpi in^2 z}.
\end{aligned}$$

Proof. By Proposition 6.2, we have

$$\begin{aligned}
\partial_k^{(h)}\left(\sum_{n=0}^{\infty} a(n)e^{2\varpi inz}\right) &= \left(\sum_{j=0}^{h}\binom{h}{j}\frac{(h+k-1)!}{(j+k-1)!}\left(\frac{-1}{4\varpi y}\right)^{h-j}D^j\right)\sum_{n=0}^{\infty} a(n)e^{2\varpi inz} \\
&= \frac{(-1)^h h!}{(4\varpi y)^h}\sum_{n=0}^{\infty} a(n)\sum_{j=0}^{h}\frac{(h+k-1)!}{(h-j)!(j+k-1)!}\frac{(-4\varpi y)^j}{j!}D^j e^{2\varpi inz} \\
&= \frac{(-1)^h h!}{(4\varpi y)^h}\sum_{n=0}^{\infty} a(n)\sum_{j=0}^{h}\binom{h+k-1}{h-j}\frac{(-4\varpi ny)^j}{j!}e^{2\varpi inz} \\
&= \frac{(-1)^h h!}{(4\varpi y)^h}\sum_{n=0}^{\infty} a(n)L_h^{k-1}(4\varpi ny)e^{2\varpi inz}.
\end{aligned}$$

For the special case $k = 1/2, 3/2$, it can be shown similarly. □

We introduce the following theta series, whose notation is based on [FK01].

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(z,\tau) := \sum_{n\in\mathbb{Z}}\exp 2\varpi i\left\{\frac{1}{2}\left(n+\frac{\varepsilon}{2}\right)^2\tau + \left(n+\frac{\varepsilon}{2}\right)\left(z+\frac{\varepsilon'}{2}\right)\right\} \quad (\varepsilon,\varepsilon'\in\mathbb{Q}), \tag{6.3}$$

$$\begin{aligned}
\theta'\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\tau) &:= \frac{\partial}{\partial z}\,\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(z,\tau)\bigg|_{z=0} \\
&= 2\varpi i\sum_{n\in\mathbb{Z}}\left(n+\frac{\varepsilon}{2}\right)\exp 2\varpi i\left\{\frac{1}{2}\left(n+\frac{\varepsilon}{2}\right)^2\tau + \frac{\varepsilon'}{2}\left(n+\frac{\varepsilon}{2}\right)\right\}. \qquad (6.4)
\end{aligned}$$

The action of the Maass–Shimura operator on (6.3) and (6.4) is described by

$$\theta_{(p)}\begin{bmatrix}\mu\\ \nu\end{bmatrix}(z) := i^{-p}(2\varpi y)^{-p/2}\sum_{n\in\mathbb{Z}+\mu}H_p(n\sqrt{2\varpi y})\exp(\varpi in^2z + 2\varpi i\nu n) \quad (\mu,\nu\in\mathbb{Q}, p\in\mathbb{Z}_{\geq 0}).$$

Proposition 6.5. For $h\in\mathbb{Z}_{\geq 0}$, it holds that

$$\begin{aligned}
\theta_{(2h)}\begin{bmatrix}\mu\\ \nu\end{bmatrix}(z) &= (-1)^h 2^{3h}\partial_{1/2}^{(h)}\left(\theta\begin{bmatrix}2\mu\\ 2\nu\end{bmatrix}(0,z)\right), \\
\theta_{(2h+1)}\begin{bmatrix}\mu\\ \nu\end{bmatrix}(z) &= -i(-1)^h 2^{3h+1}\partial_{3/2}^{(h)}\left(\frac{1}{2\varpi i}\theta'\begin{bmatrix}2\mu\\ 2\nu\end{bmatrix}(0,z)\right).
\end{aligned}$$

Proof. It follows by Proposition 6.4 and the identities (6.2). □

6.2 The L-value with Maass–Shimura operator

6.2.1 The case for E_{-p}

Let ψ be the Hecke character of $K = \mathbb{Q}(i)$ associated to $E_{-1} : y^2 = x^3 + x$. For an integral ideal $\mathfrak{a}$ of $\mathcal{O}_K$ which is prime to 4, we have

$$\psi(\mathfrak{a}) = (-1)^{(a-1)/2}(a+bi),$$

where $a+bi$ is the primary generator of $\mathfrak{a}$, that is, $a+bi$ satisfies $(a,b)\equiv(1,0),(3,2) \bmod 4$. We set $\varepsilon(a+bi) = (-1)^{(a-1)/2}$.

Lemma 6.6. An integral ideal $\mathfrak{a}$ of $\mathcal{O}_K$ which is prime to 4 is written in the form

$$\mathfrak{a} = (r + 4N - 2mi) \quad (r \in \{1, 3\},\ N, m \in \mathbb{Z}).$$

Proof. An ideal $(a+bi)$ is prime to 4 if and only if its norm a^2+b^2 is prime to 4. Therefore such an ideal $(a+bi)$ must satisfy $(a,b) \equiv (1,0), (0,1) \bmod 2$. There is nothing to prove the former case. For the latter case, it follows from $(a+bi) = (b-ai)$. □

Let $\Theta(z)$ be the theta series

$$\Theta(z) = \sum_{\lambda \in \mathcal{O}_K} q^{N_{K/\mathbb{Q}}\lambda} = \sum_{n,m \in \mathbb{Z}} q^{n^2+m^2} \in M_1(\Gamma_1(4)).$$

Proposition 6.7. We have

$$L(\psi^{2k-1}, k) = \frac{(-1)^{k-1}2^{-3}\varpi^k}{(k-1)!}\Big(\partial_1^{(k-1)}\Theta(z)|_{z=i/4} + \partial_1^{(k-1)}\Theta(z)|_{z=i/4+1/2}\Big).$$

Proof. We consider the Eisenstein series of weight 1 for $\Gamma_1(4)$

$$G_{1,\varepsilon}(z) = \lim_{s \to 0} \frac{1}{2} \sum_{n,m}{}' \frac{\varepsilon(n)}{(4mz+n)|4mz+n|^{2s}} \qquad (z \in \mathbb{H}),$$

where the prime means that summation over the terms whose denominator is not zero. By using Proposition 6.3, we have

$$\partial_1^{(k-1)} G_{1,\varepsilon}(z) = (k-1)!\left(\frac{-1}{4\varpi y}\right)^{k-1} \frac{1}{2} \sum_{n,m}{}' \frac{\varepsilon(n)(n+4m\bar{z})^{2k-1}}{|n+4mz|^{2k}}.$$

Since $G_{1,\varepsilon}(z) = \varpi/4 \cdot \Theta(z)$ (Note that $\dim M_1(\Gamma_1(4)) = 1$), it holds that

$$\begin{aligned} L(\psi^{2k-1}, k) &= \sum_{r,N,m}{}' \frac{\psi((r+4N-2mi))^{2k-1}}{|r+4N-2mi|^{2k}} \\ &= \frac{1}{2} \sum_{r,N,m}{}' \frac{\varepsilon(r+4N)(r+4N-2mi)^{2k-1}}{|r+4N+2mi|^{2k}} \\ &= \frac{1}{2} \sum_{n,m}{}' \frac{\varepsilon(n)(n-2mi)^{2k-1}}{|n+2mi|^{2k}} \\ &= \frac{(-1)^{k-1}2^{k-3}\varpi^k}{(k-1)!} \partial_1^{(k-1)}\Theta(z)|_{z=i/2}, \end{aligned} \tag{6.5}$$

Finally the identity [Köh11, p.192]

$$2\Theta(z) = \Theta\left(\frac{z}{2}\right) + \Theta\left(\frac{z+1}{2}\right)$$

yields the claim. □

Corollary 6.8. If k is an even integer, then $L(\psi^{2k-1}, k) = 0$.

Proof. For $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 4 & 0 \end{smallmatrix}\right) \in GL_2^+(\mathbb{Q})$, we have $\Theta(z)|[\gamma]_1 = -i\Theta(z)$ (*cf.* [Kob93, p.124]). By Proposition 6.1, we have

$$\partial_1^{(k-1)}\Theta(z) = i(2z)^{-2k+1}\partial_1^{(k-1)}\Theta(z)|_{z=-1/4z}.$$

Thus we obtain $\partial_1^{(k-1)}\Theta(z)|_{z=i/2} = 0$ and the colollary follows by the equality (6.5). □

Next, we write the special value $L(\psi^{2k-1}, k)$ as the square of the ∂_k-derivative of some modular form. The key is Theorem 6.9 below. For $z = x + iy \in \mathbb{H}$, we put $Q_z(n,m) = |mz - n|^2/2y$. Note that by Proposition 6.4, it holds that

$$\begin{aligned}&\partial_1^{(k-1)}\Theta(z)|_{z=i/4} + \partial_1^{(k-1)}\Theta(z)|_{z=i/4+1/2}\\ &=2\frac{(-1)^{k-1}(k-1)!}{\varpi^{k-1}}\sum_{(0,0),(1,1)} L_{k-1}^0(2\varpi Q_i(n,m))e^{-\varpi(n^2+m^2)/2},\end{aligned}$$

where $\sum_{(a,b)}$ implies that (n,m) runs over all pairs of integer which satisfy $(n,m) \equiv (a,b)$ mod 2. For simplicity, we set

$$a_{n,m} := L_{k-1}^0(2\varpi Q_i(n,m))e^{-\varpi(n^2+m^2)/2}.$$

Theorem 6.9 ([RZ93, p.7])**.** For $a \in \mathbb{Z}_{>0}$, $z \in \mathbb{H}$, $\mu, \nu \in \mathbb{Q}$ and $p, \alpha \in \mathbb{Z}_{\geq 0}$, the following identity holds.

$$\begin{aligned}&\frac{(-1)^p p!}{(\varpi y)^p}\sum_{n,m\in\mathbb{Z}} e^{2\varpi i(n\mu+m\nu)}\left(\frac{mz-n}{ay}\right)^{\alpha} L_p^{\alpha}\left(\frac{2\varpi}{a}Q_z(n,m)\right)e^{\varpi(inm-Q_z(n,m))/a}\\ &\qquad = \sqrt{2ay}(ay)^{\alpha}\theta_{(p)}\left[\begin{array}{c} a\mu \\ \nu \end{array}\right](a^{-1}z)\theta_{(p+\alpha)}\left[\begin{array}{c} \mu \\ -a\nu \end{array}\right](-a\bar{z}).\end{aligned}$$

In particular for the case $a = 1$, $\alpha = 0$, the right hand side is

$$(-1)^p\sqrt{2y}\left|\theta_{(p)}\left[\begin{array}{c} \mu \\ \nu \end{array}\right](z)\right|^2.$$

We define θ_2, θ_4 to be

$$\theta_2(z) := \theta\left[\begin{array}{c} 1 \\ 0 \end{array}\right](0,z) = \sum_{n\in\mathbb{Z}+1/2} e^{\varpi i n^2 z}, \quad \theta_4(z) := \theta\left[\begin{array}{c} 0 \\ 1 \end{array}\right](0,z) = \sum_{n\in\mathbb{Z}}(-1)^n e^{\varpi i n^2 z}.$$

Theorem 6.10. Let ψ be the Hecke character of $K = \mathbb{Q}(i)$ associated to $E_{-1} : y^2 = x^3 + x$. Then for $L(\psi^{2k-1}, s)$, we have

$$L(\psi^{2k-1}, k) = \begin{cases} \dfrac{2^{3k-9/2}\varpi^k}{(k-1)!}\left|\partial_{1/2}^{(N)}\theta_2(z)|_{z=i}\right|^2 & (k = 2N+1), \\ 0 & (k = 2N). \end{cases}$$

Proof. We apply for $p = k-1, a = 1$, $\alpha = 0, z = i$ in Theorem 6.9. By substituting $(\mu,\, \nu) = (1/2,\, 0)$, $(0,\, 1/2)$, we see that

$$\frac{(k-1)!}{\varpi^{k-1}}\left(\sum_{(0,0),(0,1),(1,1)} a_{n,m} - \sum_{(1,0)} a_{n,m}\right) = \sqrt{2}\left|\theta_{(k-1)}\left[\begin{array}{c} 1/2 \\ 0 \end{array}\right](i)\right|^2, \tag{6.6}$$

$$\frac{(k-1)!}{\varpi^{k-1}}\left(\sum_{(0,0),(1,0),(1,1)} a_{n,m} - \sum_{(0,1)} a_{n,m}\right) = \sqrt{2}\left|\theta_{(k-1)}\left[\begin{array}{c} 0 \\ 1/2 \end{array}\right](i)\right|^2. \tag{6.7}$$

Note that

$$\left|\theta_{(k-1)}\left[\begin{array}{c} 1/2 \\ 0 \end{array}\right](z)\right|^2 = \left|\theta_{(k-1)}\left[\begin{array}{c} 0 \\ 1/2 \end{array}\right](z)\right|^2.$$

By adding (6.6) and (6.7), we obtain

$$\partial_1^{(k-1)}\Theta(z)|_{z=i/4} + \partial_1^{(k-1)}\Theta(z)|_{i/4+1/2} = (-1)^{k-1}2^{3/2}\left|\theta_{(k-1)}\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(i)\right|^2.$$

Therefore the theorem follows by Proposition 6.5. □

Corollary 6.11. Under the same condition as Theorem 6.10, we have

$$L(\psi^{2k-1}, k) \geq 0.$$

6.2.2 The case for A_p

Let ψ' be the Hecke character of $K = \mathbb{Q}(\omega)$ associated to $A_1 : x^3 + y^3 = 1$, where $\omega = (-1+\sqrt{-3})/2$. For an integral ideal $\mathfrak{a}$ of $\mathcal{O}_K$ which is prime to 3, we have

$$\psi'(\mathfrak{a}) = \psi'((a+bi)) = \varepsilon'(a+bi)(a+bi),$$

where $\varepsilon' : (\mathcal{O}_K/3\mathcal{O}_K)^\times \to \mathbb{C}^\times$ is some sextic character.

Lemma 6.12. An integral ideal $\mathfrak{a}$ of $\mathcal{O}_K$ which is prime to 3 is written in the form

$$\mathfrak{a} = (r + 3(N + m\omega^2)) \quad (r \in \{1,2\}, N, m \in \mathbb{Z}),$$

Proof. A proof is the same as Lemma 6.6. □

Let $\Theta'(z)$ be the theta series

$$\Theta'(z) = \sum_{\lambda \in \mathcal{O}_K} q^{N\lambda} = \sum_{n,m} q^{n^2+nm+m^2} \in M_1(\Gamma_1(3)).$$

Proposition 6.13. We have

$$L(\psi'^{2k-1}, k) = \frac{(-1)^{k-1}2^{k-1}3^{-k/2-2}\varpi^k}{(k-1)!}\omega^{k-1}(1-\omega)\partial_1^{(k-1)}\Theta'(z)|_{z=(\omega-2)/3}.$$

Proof. Similarly for the case E_{-p}, we obtain

$$\begin{aligned} L(\psi'^{2k-1}, k) &= \frac{1}{2}\sum_{n,m}{}' \frac{\varepsilon'(n)(n+3m\omega^2)^{2k-1}}{|n+3m\omega|^{2k}} \\ &= \frac{(-1)^{k-1}2^{k-1}3^{k/2-2}\varpi^k}{(k-1)!}\partial_1^{(k-1)}\Theta'(z)|_{z=\omega}. \end{aligned}$$

For the Atkin–Lehner involution $W_3 = \begin{pmatrix} 0 & -1/\sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}$, we have $\Theta'(z)|[W_3]_1 = -i\Theta'(z)$ (*cf.* [Köh11, p.155]). By Proposition 6.1, we have

$$\partial_1^{(k-1)}\Theta'(z) = i(\sqrt{3}z)^{-2k+1}\partial_1^{(k-1)}\Theta'(z)|_{z=-1/3z}.$$

The proposition follows by substituting $z = \omega$. □

By Proposition 6.4, it holds that

$$\begin{aligned}&\partial_1^{(k-1)}\Theta(z)|_{z=(\omega-2)/3}\\ &=\frac{(-1)^{k-1}\sqrt{3}^{k-1}(k-1)!}{2^{k-1}\varpi^{k-1}}\sum_{n,m\in\mathbb{Z}} L_{k-1}^{0}(2\varpi Q_\omega(n,m))e^{2\varpi i(n^2+nm+m^2)(\omega-2)/3}.\end{aligned}$$

For simplicity, we set

$$a_{n,m}:=L_{k-1}^{0}(2\varpi Q_\omega(n,m))e^{2\varpi i(n^2+nm+m^2)(\omega-2)/3}.$$

Let $\eta(z)=q^{1/24}\prod_{n\geq 1}(1-q^n)$ be the Dedekind eta function.

Lemma 6.14. For $h,N\in\mathbb{Z}_{\geq 0}$, the following holds:

(i) $\partial_{1/2}^{(h)}\,\theta\begin{bmatrix}1/3\\-1/3\end{bmatrix}(0,z)\Big|_{z=\omega}=e^{h\varpi i/3-\varpi i/4}3^{1/4}\,\partial_{1/2}^{(h)}\eta(z)\Big|_{z=\omega}$,

(ii) $\partial_{3/2}^{(h)}\,\frac{1}{2\varpi i}\theta'\begin{bmatrix}1\\1\end{bmatrix}(z)\Big|_{z=\omega}=e^{\varpi i/2}\partial_{3/2}^{(h)}\eta(z)^3|_{z=\omega}$,

(iii) $\partial_{3/2}^{(3N+1)}\,\frac{1}{2\varpi i}\theta'\begin{bmatrix}1/3\\-1/3\end{bmatrix}(z)\Big|_{z=\omega}=e^{N\varpi i-13\varpi i/36}2^{-1}3^{5/4}\partial_{3/2}^{(3N+1)}\eta(3z)^3|_{z=\omega}$.

Proof. (i) By using identity [FK01, p.241]

$$\theta\begin{bmatrix}1/3\\1\end{bmatrix}(0,z)=e^{\varpi i/6}\eta(z),$$

we have

$$\theta\begin{bmatrix}1/3\\-1/3\end{bmatrix}(0,z)=e^{-7\varpi i/36}\theta\begin{bmatrix}1/3\\1\end{bmatrix}(0,z)=e^{-\varpi i/36}\eta\Big(\frac{z-1}{3}\Big).$$

It follows from this and Proposition 6.1.

(ii) It follows from the identity [FK01, p.289, (4.14)]

$$\theta'\begin{bmatrix}1\\1\end{bmatrix}(0,\tau)=-2\varpi\eta(\tau)^3. \tag{6.8}$$

(iii) Proposition 6.1, we have

$$\partial_{3/2}^{(3N+1)}\eta(z)^3|_{z=\omega}=0. \tag{6.9}$$

It follows from (6.8), (6.9) and the identity [FK01, p.240, (3.40)]

$$6e^{\varpi i/3}\theta'\begin{bmatrix}1/3\\1\end{bmatrix}(0,3z)=\theta'\begin{bmatrix}1\\1\end{bmatrix}(0,z/3)+3\theta'\begin{bmatrix}1\\1\end{bmatrix}(0,3z).$$

□

Theorem 6.15. Let ψ' be the Hecke character of $K = \mathbb{Q}(\omega)$ associated to $A_1 : x^3 + y^3 = 1$. Then for $L(\psi'^{2k-1}, s)$, we have

$$L(\psi'^{2k-1}, k) = \begin{cases} \dfrac{\varpi^k}{(k-1)!} 2^{2k-1} 3^{k/2-9/4} \left| \partial_{1/2}^{(3N)} \eta(z)|_{z=\omega} \right|^2 & (k = 6N+1), \\ \dfrac{\varpi^k}{(k-1)!} 2^{2k-3} 3^{k/2-11/4} \left| \partial_{3/2}^{(3N+1)} \eta(z)^3|_{z=\omega} \right|^2 & (k = 6N+2), \\ \dfrac{\varpi^k}{(k-1)!} 2^{2k-4} 3^{k/2-1/4} \left| \partial_{3/2}^{(3N+1)} \eta(3z)^3|_{z=\omega} \right|^2 & (k = 6N+4), \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. We apply for $p = k-1$, $a = 1$, $\alpha = 0$, $z = \omega$ in Theorem 6.9. By substituting $(\mu, \nu) = (1/2, 1/2)$ with multiplication by ω^2, $(\mu, \nu) = (1/6, -1/6)$ and $(-1/6, 1/6)$, we see that

$$\frac{2^{k-1}(k-1)!}{\sqrt{3}^{k-1}\varpi^{k-1}} \left(\sum_{n-m\equiv 1,2,4,5} a_{n,m} + \sum_{n-m\equiv 0,3} a_{n,m} \right) = \omega^2 \sqrt[4]{3} \left| \theta_{(k-1)} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (\omega) \right|^2, \tag{6.10}$$

$$\frac{2^{k-1}(k-1)!}{\sqrt{3}^{k-1}\varpi^{k-1}} \left(\sum_{n-m\equiv 0,1,3,4} a_{n,m} + \sum_{n-m\equiv 2,5} a_{n,m} \right) = \sqrt[4]{3} \left| \theta_{(k-1)} \begin{bmatrix} 1/6 \\ -1/6 \end{bmatrix} (\omega) \right|^2, \tag{6.11}$$

$$\frac{2^{k-1}(k-1)!}{\sqrt{3}^{k-1}\varpi^{k-1}} \left(\sum_{n-m\equiv 0,2,3,5} a_{n,m} + \sum_{n-m\equiv 1,4} a_{n,m} \right) = \sqrt[4]{3} \left| \theta_{(k-1)} \begin{bmatrix} -1/6 \\ 1/6 \end{bmatrix} (\omega) \right|^2, \tag{6.12}$$

where $\sum_{n-m\equiv a}$ implies that (n, m) runs over all pairs of integer which satisfy $n - m \equiv a \bmod 6$. Note that

$$\left| \theta_{(p)} \begin{bmatrix} \mu \\ -\nu \end{bmatrix} (z) \right|^2 = \left| \theta_{(p)} \begin{bmatrix} -\nu \\ \mu \end{bmatrix} (z) \right|^2.$$

By adding (6.10), (6.11) and (6.12), we obtain

$$L(\psi'^{2k-1}, k) = \frac{2^{-k+1} 3^{k/2-11/4} \varpi^k}{(k-1)!} \left\{ \omega^{k+1} \left| \theta_{(k-1)} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (\omega) \right|^2 + 2\omega^{k-1} \left| \theta_{(k-1)} \begin{bmatrix} 1/6 \\ -1/6 \end{bmatrix} (\omega) \right|^2 \right\}.$$

Since $L(\psi'^{2k-1}, k)$ takes a real number, it holds that

$$L(\psi'^{2k-1}, k) = \begin{cases} 0 & (k \equiv 0, 3 \bmod 6), \\ \dfrac{2^{-k+2} 3^{k/2-11/4} \varpi^k}{(k-1)!} \left| \theta_{(k-1)} \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix} (\omega) \right|^2 & (k \equiv 1, 4 \bmod 6), \\ \dfrac{2^{-k+1} 3^{k/2-11/4} \varpi^k}{(k-1)!} \left| \theta_{(k-1)} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (\omega) \right|^2 & (k \equiv 2, 5 \bmod 6). \end{cases}$$

The theorem follows by Proposition 6.5, Lemma 6.14 and the equation

$$\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (0, z) = 0.$$

□

Corollary 6.16. Under the same condition as Theorem 6.15, we have

$$L(\psi'^{2k-1}, k) \geq 0.$$

Chapter 7

Recurrence formula for the algebraic part

7.1 On the Cohen–Kuznetsov series

The differential operator D does not preserve modularity, but it does preserve holomorphy. On the other hand, the Maass–Shimura operator ∂_k preserves modularity, but does not preserve holomorphy. We introduce an operator that preserves the properties of both modularity and holomorphy. Let us denote the Ramanujan–Serre operator by

$$\vartheta_k = D - \frac{k}{12} E_2.$$

Here, $E_2(z) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n$ is the Eisenstein series of weight 2. This Eisenstein series is not a modular form, but the function $E_2^*(z) = E_2(z) - 3/\varpi y$ is a non-holomorphic modular form. Since the Ramanujan–Serre operator is also expressed as $\vartheta_k = \partial_k - kE_2^*/12$, we see that ϑ_k maps a modular form of weight k to a modular form of weight $k+2$. We sometimes drop the subscript k of ϑ_k if it is clear the weight of a modular form on which ϑ acts.

To express the difference between the operators D, ∂_k and ϑ_k, Rodríguez-Villegas and Zagier have introduced the Cohen–Kuznetsov series

$$f_D(z, X) \coloneqq \sum_{n=0}^{\infty} \frac{D^n f(z)}{(k)_n} \frac{X^n}{n!} \quad (z \in \mathbb{H},\, X \in \mathbb{C},\, f \in M_k(\Gamma))$$

and modified Cohen–Kuznetsov series

$$f_\partial(z, X) \coloneqq \sum_{n=0}^{\infty} \frac{\partial_k^{(n)} f(z)}{(k)_n} \frac{X^n}{n!},$$

where $(k)_n = k(k+1)\cdots(k+n-1)$ is the Pochhammer symbol.

Proposition 7.1. The following holds:

$$f_\partial(z, X) = e^{-X/4\varpi y} f_D(z, X).$$

Proof. By direct computation, we have

$$e^{-X/4\varpi y} f_D(z, X) = \sum_{n=0}^{\infty} \left(\sum_{\ell=0}^{n} \binom{n}{\ell} \frac{(n+k-1)!}{(\ell+k-1)!} D^\ell f(z) \right) \frac{X^n}{(k)_n n!}.$$

The claim follows from Proposition 6.2 □

Proposition 7.2. Let $f \in M_k(\Gamma)$. For all $z \in \mathbb{H}, X \in \mathbb{C}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, it follows that

$$f_D\left(\frac{az+b}{cz+d}, \frac{X}{(cz+d)^2}\right) = (cz+d)^k \exp\left(\frac{c}{cz+d}\frac{X}{2\varpi i}\right) f_D(z, X), \tag{7.1}$$

$$f_\partial\left(\frac{az+b}{cz+d}, \frac{X}{(cz+d)^2}\right) = (cz+d)^k f_\partial(z, X). \tag{7.2}$$

Proof. The equation (7.2) follows from the fact that $\partial_k^{(n)} f$ transforms like a modular form of weight $k+2n$. For the equation (7.1), by using Proposition 7.1, we have

$$\begin{aligned} f_D\left(\frac{az+b}{cz+d}, \frac{X}{(cz+d)^2}\right) &= \exp\left(\frac{X}{4\varpi y}\frac{c\bar{z}+d}{cz+d}\right) f_\partial\left(\frac{az+b}{cz+d}, \frac{X}{(cz+d)^2}\right) \\ &= (cz+d)^k \exp\left(\frac{X}{4\varpi y}\frac{c\bar{z}+d}{cz+d}\right) f_\partial(z, X) \\ &= (cz+d)^k \exp\left(\frac{X}{4\varpi y}\left(\frac{c\bar{z}+d}{cz+d} - 1\right)\right) f_D(z, X) \\ &= (cz+d)^k \exp\left(\frac{c}{cz+d}\frac{X}{2\varpi i}\right) f_D(z, X). \end{aligned}$$

□

A series such as the Cohen–Kuznetsov series for the Ramanujan–Serre operator is not defined in the same way. We define

$$f_\vartheta(z, X) := e^{-E_2^*(z)X/12} f_\partial(z, X). \tag{7.3}$$

Then, expansion coefficients of $f_\vartheta(z, X)$ satisfy a certain recurrence relation.

Proposition 7.3 ([RZ95, p.12])**.** Let $f \in M_k(\Gamma)$. Then the series $f_\vartheta(z, X)$ has the expansion

$$f_\vartheta(z, X) = \sum_{n=0}^{\infty} \frac{F_n(z)}{(k)_n}\frac{X^n}{n!},$$

where $F_n \in M_{k+2n}(\Gamma)$ is the modular form that is defined by the following recurrence formula:

$$F_{n+1} = \vartheta_{k+2n} F_n - \frac{n(n+k-1)}{144} E_4 F_{n-1}. \tag{7.4}$$

The initial condition is $F_0 = f$, $F_1 = \vartheta_k f$.

Proof. Proposition 7.2 shows the function F_n is a modular form of weight $k+2n$. By the definition (7.3) and Proposition 7.1, we have

$$f_\vartheta(z, X) = e^{-E_2(z)X/12} f_D(z, X). \tag{7.5}$$

Expanding the right-hand side of (7.5), we obtain

$$F_n(z) = \sum_{\ell=0}^{n} \frac{n!}{\ell!}\binom{n+k-1}{n-\ell}\left(-\frac{E_2(z)}{12}\right)^{n-\ell} D^\ell f(z). \tag{7.6}$$

We can see that the function (7.6) satisfies the recurrence formula (7.4) by using the equation [Bru+08, Proposition 15]

$$D\left(-\frac{E_2(z)}{12}\right) - \left(-\frac{E_2(z)}{12}\right)^2 = \frac{E_4(z)}{144}.$$

□

If a CM point z_0 satisfies $E_2^*(z_0) = 0$, then $f_\partial(z_0, X) = f_\vartheta(z_0, X)$ by (7.3). Therefore by Proposition 7.3, we see that

$$\partial_k^{(n)} f(z)|_{z=z_0} = F_n(z_0),$$

where F_n is the modular form that is defined by the recurrence formula (7.4).

7.2 Recurrence formula

7.2.1 The case for E_{-p}

We apply Proposition 7.3 for $f = \theta_2, \Gamma = \Gamma(2)$. The graded ring $\oplus_{k\in\frac{1}{2}\mathbb{Z}} M_k(\Gamma(2))$ is isomorphic to $\mathbb{C}[\theta_2, \theta_4]$ as $\mathbb{C}$-algebra (*cf.* [Bru+08, p.28-29]). Since θ_2 and θ_4 is algebraically independent over $\mathbb{C}$, we sometimes regard θ_2 and θ_4 as indeterminates and $\mathbb{C}[\theta_2, \theta_4]$ as the polynomial ring in two variables over $\mathbb{C}$.

Lemma 7.4. We have

$$\vartheta\theta_2 = \frac{1}{12}\theta_2\theta_4^4 + \frac{1}{24}\theta_2^5, \quad \vartheta\theta_4 = -\frac{1}{12}\theta_2^4\theta_4 - \frac{1}{24}\theta_4^5.$$

Proof. It follows from the fact that $\vartheta\theta_2^4$ and $\vartheta\theta_4^4$ are of weight 4 and the ring $M_4(\Gamma(2))$ is generated by θ_2^4, θ_4^4. □

By Lemma 7.4, the Ramanujan–Serre operator ϑ acts on $\mathbb{C}[\theta_2, \theta_4]$ as

$$\vartheta = \left(\frac{1}{12}\theta_2\theta_4^4 + \frac{1}{24}\theta_2^5\right)\frac{\partial}{\partial\theta_2} - \left(\frac{1}{12}\theta_2^4\theta_4 + \frac{1}{24}\theta_4^5\right)\frac{\partial}{\partial\theta_4}.$$

Lemma 7.5. The following holds:

$$\theta_2(i) = 2^{-1/4}\varpi^{-1/2}\Omega_E^{1/2}.$$

Proof. The lemma holds from the identity $\theta_2(z) = 2\eta(2z)^2/\eta(z)$ (For example, see [Bru+08, p.28-29]) and well-known formula:

$$\eta(i) = \frac{\Gamma(1/4)}{2\varpi^{3/4}}, \quad \eta(2i) = \frac{\Gamma(1/4)}{2^{11/8}\varpi^{3/4}}.$$

□

Theorem 7.6. We define the algebraic part of $L(\psi^{2k-1}, k)$ to be

$$L_{E,k} = \frac{2^{k+1}3^{k-1}\varpi^{k-1}(k-1)!}{\Omega_E^{2k-1}}L(\psi^{2k-1}, k).$$

Then $L_{E,k}$ is the square of a rational integer and

$$\sqrt{L_{E,k}} = \begin{cases} |f_N(0)| & (k = 2N+1), \\ 0 & (k = 2N), \end{cases}$$

where $f_n(t) \in \mathbb{Z}[t]$ is the polynomial that is defined by the recurrence formula

$$f_{n+1}(t) = (4n+1)(2t+3)f_n(t) - 12(t+1)(t+2)f_n'(t) - 2n(2n-1)(t^2+3t+3)f_n(t).$$

The initial condition is $f_0(t) = 1,\ f_1(t) = 2t+3$.

Proof. By Proposition 7.3 and Lemma 7.4, we have $\partial_{1/2}^{(n)}\theta_2(z)|_{z=i} = F_n(i)$, where F_n is the modular form that is defined by the recurrence formula

$$F_{n+1} = \left(\frac{1}{12}\theta_2\theta_4^2 + \frac{1}{24}\theta_2^5\right)\frac{\partial F_n}{\partial\theta_2} - \left(\frac{1}{12}\theta_2^4\theta_4 + \frac{1}{24}\theta_4^5\right)\frac{\partial F_n}{\partial\theta_4} - \frac{n(n-1/2)}{144}E_4F_{n-1}. \tag{7.7}$$

We set $f_n = 24^nF_n/\theta_2^{4n+1}$, which has degree 0. Then we can rewrite the recurrence formula (7.7) as follows:

$$f_{n+1} = (4n+1)\frac{\theta_2^4+2\theta_4^4}{\theta_2^4}f_n + \frac{\theta_2^4+2\theta_4^4}{\theta_2^4}\frac{\partial f_n}{\partial\theta_2} - \frac{2\theta_2^4\theta_4+\theta_4^5}{\theta_2^4}\frac{\partial f_n}{\partial\theta_4} - 2n(2n-1)\frac{E_4}{\theta_2^8}f_{n-1}. \tag{7.8}$$

Moreover we set $t = (\theta_4^4-\theta_2^4)/\theta_2^4$ which satisfies $t(i) = 0$. Note that $E_4 = \theta_2^8+\theta_2^4\theta_4^4+\theta_4^8$. Then the recurrence formula (7.8) transforms

$$f_{n+1}(t) = (4n+1)(2t+3)f_n(t) - 12(t+1)(t+2)f_n'(t) - 2n(2n-1)(t^2+3t+3)f_n(t).$$

The initial condition is $f_0(t) = 1$, $f_1(t) = 2t+3$. Therefore, by Lemma 7.5, we obtain

$$\left|\partial_{1/2}^{(N)}\theta_2(z)|_{z=i}\right|^2 = 2^{-4k+7/2}3^{-k+1}\varpi^{-2k+1}\Omega_E^{2k-1}|f_N(0)|^2.$$

□

7.2.2 The case for A_p

First we consider the case for $k = 6N+1$ (The case for $k = 6N+2$ is almost the same). We apply Proposition 7.3 for $f = \eta, \Gamma = \Gamma(1)$. The graded ring $\oplus_{k\in\mathbb{Z}}M_k(\Gamma(1))$ is isomorphic to $\mathbb{C}[E_4,E_6]$ as $\mathbb{C}$-algebra. Since E_4 and E_6 are algebraically independent over $\mathbb{C}$, we sometimes regard E_4 and E_6 as indeterminates and $\mathbb{C}[E_4,E_6]$ as the polynomial ring in two variables over $\mathbb{C}$. We denote by $\frac{\partial}{\partial E_4}$ and $\frac{\partial}{\partial E_6}$ the derivative with respect to formal variables E_4 and E_6. We take a sufficiently small neighborhood D of ω so that $E_6^{1/3}$ can be defined. (Note that $E_6(\omega) \neq 0$.) In the following, we restrict the domain of functions in $\mathbb{C}[E_4,E_6,E_6^{1/3},E_6^{-1},\eta]$ to D.

Lemma 7.7. We have

$$\vartheta E_4 = -\frac{1}{3}E_6, \quad \vartheta E_6 = -\frac{1}{2}E_4^2, \quad \vartheta\eta = 0.$$

Proof. The proof is the same as Lemma 7.4. □

By the above lemma, the Ramanujan–Serre operator ϑ acts on $\mathbb{C}[E_4,E_6]$ as

$$\vartheta = -\frac{E_6}{3}\frac{\partial}{\partial E_4} - \frac{E_4^2}{2}\frac{\partial}{\partial E_6}. \tag{7.9}$$

The derivatives $\frac{\partial}{\partial E_4}$ and $\frac{\partial}{\partial E_6}$ on $\mathbb{C}[E_4,E_6]$ are uniquely extended on $\mathbb{C}[E_4,E_6,E_6^{-1},E_6^{1/3},\eta]$ satisfying the following:

$$\frac{\partial}{\partial E_6}E_6^{-1} = -E_6^{-2}, \quad \frac{\partial}{\partial E_6}E_6^{1/3} = \frac{1}{3}E_6^{-1}E_6^{1/3}.$$

Next we consider the case for $k = 6N+4$. We apply Proposition 7.3 for $f = \eta_3, \Gamma = \Gamma_0(3)$, where $\eta_3(z) = \eta(3z)^3$. It is known that the graded ring $\oplus_{k\in\mathbb{Z}}M_k(\Gamma_0(3))$ is isomorphic to $\mathbb{C}[C,\alpha,\beta]/(\alpha^2-C\beta) \cong \mathbb{C}[C,C^{-1},\alpha]$ (*cf.* [Sud11]) as $\mathbb{C}$-algebra, where

$$C = \frac{1}{2}(3E_2(3z)-E_2(z)), \quad \alpha = \frac{1}{240}(E_4(z)-E_4(3z)),$$

$$\beta = \frac{1}{12}\left\{\frac{1}{504}(E_6(3z) - E_6(z)) - C\alpha\right\}.$$

Since C and α are algebraically independent over $\mathbb{C}$, we sometimes regard C and α as indeterminates and $\mathbb{C}[C,\alpha]$ as the polynomial ring in two variables over $\mathbb{C}$. In the following, we consider the extension $\mathbb{C}[C, C^{-1}, \alpha, \eta_3]$ of $\mathbb{C}[C,\alpha]$.

Lemma 7.8. We have

$$\vartheta C = -\frac{1}{6}C^2 + 18\alpha, \quad \vartheta\alpha = \frac{2}{3}C\alpha + 9C^{-1}\alpha^2.$$

Proof. The proof is the same as Lemma 7.4. □

Similarly in the case for $k = 6N+1$, by Lemma 7.8 the Ramanujan–Serre operator ϑ acts on $\mathbb{C}[C, C^{-1}, \alpha, \eta_3]$ as

$$\vartheta = \left(-\frac{1}{6}C^2 + 18\alpha\right)\frac{\partial}{\partial C} + \left(\frac{2}{3}C\alpha + 9C^{-1}\alpha^2\right)\frac{\partial}{\partial \alpha}.$$

Lemma 7.9. The following holds:

$$|\eta(\omega)| = \frac{3^{3/8}\Omega_A^{1/2}}{2^{1/2}\varpi^{1/2}}, \quad |\eta_3(\omega)| = \frac{\Omega_A^{3/2}}{2^{3/2}3^{1/8}\varpi^{3/2}}, \quad E_6(\omega) = \frac{3^6\Omega_A^6}{2^3\varpi^6}, \quad C(\omega) = \frac{3\Omega_A^2}{\varpi^2}.$$

Proof. It can be shown in the same way as Lemma 7.5. □

Theorem 7.10. We define the algebraic part of $L(\psi'^{2k-1}, k)$ to be

$$L_{A,k} = 3\nu\left(\frac{2\varpi}{3\sqrt{3}\Omega_A^2}\right)^{k-1}\frac{(k-1)!}{\Omega_A}L(\psi'^{2k-1}, k),$$

where $\nu = 2$ if $k \equiv 2 \bmod 6$, $\nu = 1$ otherwise. Then $L_{A,k}$ is the square of a rational integer and

$$\sqrt{L_{A,k}} = \begin{cases} |x_{3N}(0)| & (k = 6N+1), \\ |y_{3N}(0)| & (k = 6N+2), \\ |z_{3N+1}(0)| & (k = 6N+4), \\ 0 & (\text{otherwise}), \end{cases}$$

where $x_n(t), y_n(t), z_n(t) \in \mathbb{Z}[t]$ are polynomials that is defined by the following recurrece formulas

$$\begin{aligned} x_{n+1}(t) &= -2(1-8t^3)x_n'(t) - 8nt^2x_n(t) - n(2n-1)tx_{n-1}(t), \\ y_{n+1}(t) &= -2(1-8t^3)y_n'(t) - 8nt^2y_n(t) - n(2n+1)ty_{n-1}(t), \\ z_{n+1}(t) &= -(t-1)(9t-1)z_n'(t) + \{(6t-2)n+2\}z_n(t) - 2n(2n+1)tz_{n-1}(t). \end{aligned}$$

The initial conditions are

$$\begin{aligned} x_0(t) &= 1, \quad x_1(t) = 0, \\ y_0(t) &= 1, \quad y_1(t) = 0, \\ z_0(t) &= 1/2, \quad z_1(t) = 1. \end{aligned}$$

Proof. Since the proof for the case $k = 6N+2$ is the same for $k = 6N+1$, we prove for the case $k = 6N+1, 6N+4$.

First we prove for $k = 6N+1$. By Proposition 7.3 and the equation (7.9), we have $\partial_{1/2}^{(n)}\eta(z)|_{z=\omega} = X_n(\omega)$, where X_n is the modular form that is defined by the recurrence formula

$$X_{n+1} = -\frac{E_6}{3}\frac{\partial X_n}{\partial E_4} - \frac{E_4^2}{2}\frac{\partial X_n}{\partial E_6} - \frac{n(n-1/2)}{144}E_4 X_{n-1}. \tag{7.10}$$

We set $x_n = 12^n X_n/\eta E_6^{n/3}$ and $t = E_4 E_6^{-2/3}/2$ which satisfies $t(\omega) = 0$. Then we can rewrite the recurrence formula (7.10) as follows:

$$x_{n+1}(t) = -2(1-8t^3)x_n'(t) - 8nt^2 x_n(t) - n(2n+1)t x_{n-1}(t).$$

The initial condition is $x_0(t) = 1$, $x_1(t) = 0$. Therefore, by Lemma 7.9, we obtain

$$\left|\partial_{1/2}^{(3N)}\eta(z)|_{z=\omega}\right|^2 = \frac{\Omega_A^{2k-1}}{\varpi^{2k-1}} 2^{-3k+2} 3^{k-1/4} |x_{3N}(0)|^2.$$

Next we prove for $k = 6N+4$. We set $\eta_3(z) = \eta(3z)^3$. We have $\partial_{3/2}^{(n)}\eta_3(z)|_{z=\omega} = Z_n(\omega)$, where Z_n is the modular form that is defined by the recurrence formula

$$Z_{n+1} = \left(-\frac{1}{6}C^2 + 18\alpha\right)\frac{\partial Z_n}{\partial C} + \left(\frac{2}{3}C\alpha + 9C^{-1}\alpha^2\right)\frac{\partial Z_n}{\partial \alpha} - \frac{n(n+1/2)}{144}E_4 Z_{n-1}. \tag{7.11}$$

We set $z_n = 2^{3n-1}Z_n/\eta_3 C^n, t = (1+216C^{-2}\alpha)/9$, which satisfies $t(\omega) = 0$. Then we can rewrite the recurrence formula (7.11) as follows:

$$z_{n+1}(t) = -(t-1)(9t-1)z_n'(t) + \{(6t-2)n+2\}z_n(t) - 2n(2n+1)t z_{n-1}(t).$$

The initial condition is $z_0(t) = 1/2$, $z_1(t) = 1$. Therefore, by Lemma 7.9, we obtain

$$\left|\partial_{3/2}^{(3N+1)}\eta_3(z)|_{z=\omega}\right| = \frac{\Omega_A^{2k-1}}{\varpi^{2k-1}} 2^{-3k+5} 3^{k-9/4} |z_{3N+1}(0)|^2.$$

□

Bibliography

[BS65] B. J. Birch and H. P. F. Swinnerton-Dyer. "Notes on elliptic curves. II". In: *J. Reine Angew. Math.* 218 (1965), pp. 79–108.

[BCP97] W. Bosma, J. Cannon, and C. Playoust. "The Magma algebra system. I. The user language". In: vol. 24. 3-4. Computational algebra and number theory (London, 1993). 1997, pp. 235–265.

[Bre+01] C. Breuil et al. "On the modularity of elliptic curves over $\mathbb{Q}$: wild 3-adic exercises". In: *J. Amer. Math. Soc.* 14.4 (2001), pp. 843–939.

[Bru+08] J. H. Bruinier et al. *The 1-2-3 of modular forms.* Universitext. Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004, Edited by Kristian Ranestad. Springer-Verlag, Berlin, 2008, pp. x+266.

[Cho19] J. Choi. "On the 2-adic valuations of central L-values of elliptic curves". In: *J. Number Theory* 204 (2019), pp. 405–422.

[CW77] J. Coates and A. Wiles. "On the conjecture of Birch and Swinnerton-Dyer". In: *Invent. Math.* 39.3 (1977), pp. 223–251.

[Coa+14] J. Coates et al. "On the 2-part of the Birch–Swinnerton-Dyer conjecture for elliptic curves with complex multiplication". In: *Münster J. Math.* 7.1 (2014), pp. 83–103.

[Coa+15] J. Coates et al. "Quadratic twists of elliptic curves". In: *Proc. Lond. Math. Soc. (3)* 110.2 (2015), pp. 357–394.

[Dam70] R. M. Damerell. "L-functions of elliptic curves with complex multiplication. I". In: *Acta Arith.* 17 (1970), pp. 287–301.

[DV18] S. Dasgupta and J. Voight. "Sylvester's problem and mock Heegner points". In: *Proc. Amer. Math. Soc.* 146.8 (2018), pp. 3257–3273.

[DD10] T. Dokchitser and V. Dokchitser. "On the Birch-Swinnerton-Dyer quotients modulo squares". In: *Ann. of Math. (2)* 172.1 (2010), pp. 567–596.

[Dri73] V. G. Drinfel'd. "Two theorems on modular curves". In: *Funkcional. Anal. i Priložen.* 7.2 (1973), pp. 83–84.

[FK01] H. M. Farkas and I. Kra. *Theta constants, Riemann surfaces and the modular group.* Vol. 37. Graduate Studies in Mathematics. An introduction with applications to uniformization theorems, partition identities and combinatorial number theory. American Mathematical Society, Providence, RI, 2001, pp. xxiv+531.

[GS81] C. Goldstein and N. Schappacher. "Séries d'Eisenstein et fonctions L de courbes elliptiques à multiplication complexe". In: *J. Reine Angew. Math.* 327 (1981), pp. 184–218.

[Gre99] R. Greenberg. "Iwasawa theory for elliptic curves". In: *Arithmetic theory of elliptic curves (Cetraro, 1997).* Vol. 1716. Lecture Notes in Math. Springer, Berlin, 1999, pp. 51–144.

[GZ86] B. H. Gross and D. B. Zagier. "Heegner points and derivatives of L-series". In: *Invent. Math.* 84.2 (1986), pp. 225–320.

[Iwa19] K. Iwasawa. *Hecke's L-functions.* SpringerBriefs in Mathematics. Spring, 1964, Lectures at Princeton University, With a foreword by John Coates and Masato Kurihara. Springer, Singapore, 2019, pp. xi+93.

[Kat04] K. Kato. "p-adic Hodge theory and values of zeta functions of modular forms". In: *Astérisque* 295 (2004). Cohomologies p-adiques et applications arithmétiques. III, pp. ix, 117–290.

[Kat76] N. M. Katz. "p-adic interpolation of real analytic Eisenstein series". In: *Ann. of Math. (2)* 104.3 (1976), pp. 459–571.

[Kez21] Y. Kezuka. "Tamagawa number divisibility of central L-values of twists of the Fermat elliptic curve". In: *J. Théor. Nombres Bordeaux* 33.3, part 2 (2021), pp. 945–970.

[Kob93] N. Koblitz. *Introduction to elliptic curves and modular forms.* Second. Vol. 97. Graduate Texts in Mathematics. Springer-Verlag, New York, 1993, pp. x+248.

[Köh11] G. Köhler. *Eta products and theta series identities.* Springer Monographs in Mathematics. Springer, Heidelberg, 2011, pp. xxii+621.

[Lem00] F. Lemmermeyer. *Reciprocity laws.* Springer Monographs in Mathematics. From Euler to Eisenstein. Springer-Verlag, Berlin, 2000, pp. xx+487.

[Lox77] J. H. Loxton. "On the determination of Gauss sums". In: *Séminaire Delange-Pisot-Poitou, 18e année: 1976/77, Théorie des nombres, Fasc. 2.* Secrétariat Math., Paris, 1977, Exp. No. 27, 12.

[Man72] J. I. Manin. "Parabolic points and zeta functions of modular curves". In: *Izv. Akad. Nauk SSSR Ser. Mat.* 36 (1972), pp. 19–66.

[Nek09] J. Nekovář. "On the parity of ranks of Selmer groups. IV". In: *Compos. Math.* 145.6 (2009). With an appendix by Jean-Pierre Wintenberger, pp. 1351–1359.

[Nom22a] K. Nomoto. *Lower bound for the* 2*-adic valuations of central L-values of elliptic curves with complex multiplication.* 2022.

[Nom22b] K. Nomoto. "The rank of a CM elliptic curve and a recurrence formula". In: *J. Number Theory* 238 (2022), pp. 60–81.

[OS21] Y. Onishi and F. Sairaiji. "Arithmetic over the Gaussian number field on a certain family of elliptic curves with complex multiplication". In: *RIMS Kôkyûroku Bessatsu* (Apr. 2021), pp. 1–22.

[Qiu03] D. Qiu. "On p-adic valuations of $L(1)$ of elliptic curves with CM by $\sqrt{-3}$". In: *Proc. Roy. Soc. Edinburgh Sect. A* 133.6 (2003), pp. 1389–1407.

[QZ02a] D. Qiu and X. Zhang. "L-series and their 2-adic valuations at $s = 1$ attached to CM elliptic curves". In: *Acta Arith.* 103.1 (2002), pp. 79–95.

[QZ02b] D. Qiu and X. Zhang. "Elliptic curves with CM by $\sqrt{-3}$ and 3-adic valuations of their L-series". In: *Manuscripta Math.* 108.3 (2002), pp. 385–397.

[RZ93] F. Rodríguez-Villegas and D. Zagier. "Square roots of central values of Hecke L-series". In: Oxford Sci. Publ. (1993), pp. 81–99.

[RZ95] F. Rodríguez-Villegas and D. Zagier. "Which primes are sums of two cubes?" In: CMS Conf. Proc. 15 (1995), pp. 295–306.

[Rub91] K. Rubin. “The “main conjectures” of Iwasawa theory for imaginary quadratic fields”. In: *Invent. Math.* 103.1 (1991), pp. 25–68.

[Rub99] K. Rubin. “Elliptic curves with complex multiplication and the conjecture of Birch and Swinnerton-Dyer”. In: *Arithmetic theory of elliptic curves (Cetraro, 1997).* Vol. 1716. Lecture Notes in Math. Springer, Berlin, 1999, pp. 167–234.

[Sat86] P. Satgé. “Groupes de Selmer et corps cubiques”. In: *J. Number Theory* 23.3 (1986), pp. 294–317.

[ST68] J.-P. Serre and J. Tate. “Good reduction of abelian varieties”. In: *Ann. of Math. (2)* 88 (1968), pp. 492–517.

[Sha87] E. de Shalit. *Iwasawa theory of elliptic curves with complex multiplication.* Vol. 3. Perspectives in Mathematics. Academic Press, Inc., Boston, MA, 1987, pp. x+154.

[Sil86] J. H. Silverman. *The arithmetic of elliptic curves.* Vol. 106. Graduate Texts in Mathematics. Springer-Verlag, New York, 1986, pp. xii+400.

[Sil94] J. H. Silverman. *Advanced topics in the arithmetic of elliptic curves.* Vol. 151. Graduate Texts in Mathematics. Springer-Verlag, New York, 1994, pp. xiv+525.

[SU14] C. Skinner and E. Urban. “The Iwasawa main conjectures for GL_2”. In: *Invent. Math.* 195.1 (2014), pp. 1–277.

[Sud11] T. Suda. *Explicit structure of the graded ring of modular forms.* 2011. arXiv: 1111.4777 [math.NT].

[Tat67] J. T. Tate. “Fourier analysis in number fields, and Hecke’s zeta-functions”. In: *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965).* Thompson, Washington, D.C., 1967, pp. 305–347.

[Tat95] J. Tate. “On the conjectures of Birch and Swinnerton-Dyer and a geometric analog”. In: *Séminaire Bourbaki, Vol. 9.* Soc. Math. France, Paris, 1995, Exp. No. 306, 415–440.

[TW95] R. Taylor and A. Wiles. “Ring-theoretic properties of certain Hecke algebras”. In: *Ann. of Math. (2)* 141.3 (1995), pp. 553–572.

[VM74] M. M. Višik and J. I. Manin. “p-adic Hecke series of imaginary quadratic fields”. In: *Mat. Sb. (N.S.)* 95(137) (1974), pp. 357–383, 471.

[Wei56] A. Weil. “On a certain type of characters of the idèle-class group of an algebraic number-field”. In: *Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955.* Science Council of Japan, Tokyo, 1956, pp. 1–7.

[Wil95] A. Wiles. “Modular elliptic curves and Fermat’s last theorem”. In: *Ann. of Math. (2)* 141.3 (1995), pp. 443–551.

[Yin22] H. Yin. “On the 8 case of the Sylvester conjecture”. In: *Trans. Amer. Math. Soc.* 375.4 (2022), pp. 2705–2728.

[ZK87] D. Zagier and G. Kramarz. “Numerical investigations related to the L-series of certain elliptic curves”. In: *J. Indian Math. Soc. (N.S.)* 52 (1987), 51–69 (1988).

[Zha97] C. Zhao. “A criterion for elliptic curves with lowest 2-power in $L(1)$”. In: *Math. Proc. Cambridge Philos. Soc.* 121.3 (1997), pp. 385–400.

[Zha01] C. Zhao. “A criterion for elliptic curves with second lowest 2-power in $L(1)$”. In: *Math. Proc. Cambridge Philos. Soc.* 131.3 (2001), pp. 385–404.

[Zha03] C. Zhao. “A criterion for elliptic curves with lowest 2-power in $L(1)$. II”. In: *Math. Proc. Cambridge Philos. Soc.* 134.3 (2003), pp. 407–420.